DISCUSSION PAPER

August 2012 ■ RFF DP 12-38

The Equilibrium Price Path of Timber in the Absence of Replanting

Stephen W. Salant

1616 P St. NW
Washington, DC 20036
202-328-5000 www.rff.org

RESOURCES FOR THE FUTURE

The Equilibrium Price Path of Timber in the Absence of Replanting[1]

Stephen W. Salant[2]

August 16, 2012

[1]I would like to express my thanks to Heywood Fleisig for renewing my interest in this subject and to Jon Faust, Kenneth Lyon, Roger Sedjo, and an anonymous referee for comments. I began this research while a Gilbert White Fellow at Resources for the Future. As always, Gérard Gaudet and Michel Moreaux provided useful comments. I alone am responsible for errors and omissions.

[2]Department of Economics, University of Michigan, and Resources for the Future (RFF), *ssalant@umich.edu*

Abstract

The forestry literature has sought to describe competitive equilibria by first solving social planning problems. This "indirect" approach may cease to be useful in determining market equilibrium if the government intervenes. The equilibrium price path of timber is characterized *directly* here under the assumption that once a site is cleared, the site is used for some other purpose of exogenous value. While extreme, this assumption permits us to show that familiar Herfindahl results from the Hotelling literature extend to forestry economics: if differing in age, older trees are harvested first; if different in site value, trees on more valuable land are harvested first. As trees of the same vintage (or site value) are harvested, the timber price may decline during intervals when wood volume grows faster than the rate of interest. As the concluding section suggests, some of these results reappear in special cases of the model with replanting.

Keywords: trees, forestry, Hotelling, Herfindahl, Faustmann, U-shaped price path

JEL Classification Numbers: Q23, Q30

1 Introduction

The earliest contribution to the theory of exhaustible resources (Gray, 1914) investigated the behavior of a wealth-maximizing extractor selling at an exogenous (and constant) price. Seventeen years later, Hotelling (1931) showed how the price of the extracted resource would be determined over time, and subsequent contributions to the literature have continued either to endogenize the price path or to determine it implicitly as the marginal utility path in the solution to the social planning problem.[1]

The literature on forestry began in much the same way. The price of timber was treated as an exogenous constant and the behavior of a wealth-maximizing extractor was deduced. The discussion centered on whether or not the extractor should take account of the value of the site after the tree currently occupying it was harvested. Jevons, Wicksell, and Fisher implicitly treated the site as without value while Faustmann (1849) assumed that after any tree was harvested a new tree would be planted in its place. Assuming the site had no value, Jevons and others concluded that a tree should be cut when the percentage rate of increase of its wood volume (and hence the monetary value of its wood content) fell to the rate of interest.

Samuelson (1976) pointed out that an extractor cannot be maximizing his wealth if he disregards the value of the site when deciding when to harvest a tree sitting on it. He also showed that Faustmann's correct approach is equivalent to the problem of cutting a tree at the optimal age while expecting to receive a financial bequest equal in value to the discounted value of

[1]For a modern survey of this literature, see Gaudet (2007).

using that site optimally in subsequent rotations. Since this bequest value is positive, Faustmann recommended cutting the tree *before* its rate of growth declined to the rate of interest. Indeed, if the site when cleared could be used for something even more valuable than subsequent rotations, then the initial stand should be harvested even earlier than Faustmann's recommendation since the bequest value of the alternate use would, by assumption, be even larger.

A half century later, economists finally followed Hotelling's lead and began to relax the assumption of an exogenous constant price in forestry models. Adopting Faustmann's assumption that a new tree would be replanted whenever a tree was harvested, this new strand of the literature attempted to determine how a planner would manage a plot of land over an infinite horizon if the trees on it had an arbitrary initial age distribution and the planner sought to maximize the discounted sum of utility derived from consuming the wood from harvested trees.

Lyon (1981) and Mitra and Wan (1985) were among the first to study this problem theoretically, and Salo and Tahvonen (2002, 2003) have further illuminated it. A central question has been whether the optimal plan eventually converges to a steady state where consumption is maintained and the age distribution recurs in successive periods or whether consumption (and, implicitly, timber prices) endlessly oscillates.[2]

The social planning problem was also investigated for specific functions and parameters by solving the optimization problem numerically on a computer. Lyon and Sedjo (1983) pioneered this approach with their "supply-

[2] This literature is ably summarized in Amacher et al. (2009).

potential optimal control model" (SPOC), which they subsequently extended in Sedjo and Lyon (1990). This extension, dubbed the "timber supply model" (TSM), continued to utilize the approach of maximizing net discounted surplus and then examining the properties of the implicit path of timber prices as reflected in the marginal utility path of timber use. In contrast, none of the more theoretical approaches focused on the path of this implicit price.

While the surplus-maximizing approach is unquestionably valuable, its shortcomings in illuminating market equilibria are well known. Stokey et al. (1989, chapter 18) refer to such analyses as "the indirect approach." As they point out, "in the presence of taxes, externalities, or other distortions, this type of attack fails."[3]

Given this, analyzing equilibrium conditions directly is often a useful complement to the programming approach. We explore this "direct" approach here. To do so, we simplify the analysis by assuming that once a tree is cut, it is infeasible to plant another tree in its place; instead, the site is used for something else the value of which is specified exogenously.

Although unrealistic, this assumption permits us to clarify the relationship between the forestry model and the Hotelling model. Consider the benchmark case where all trees of the same age sit on sites of the same value. Then if the price were constant, it is wealth-maximizing to harvest every tree at the same date (assuming the payoff function is strictly concave in the harvesting date). But, assuming that demand is downward-sloping and the

[3]While it is sometimes possible even then to devise a maximization problem for a pseudo-economy that is maximized by competitive equilibrium distorted by certain taxes, Stokey et al. point out that "the indirect psuedo-economy device. . . has very limited applicability."

exogenously specified price induces strictly positive demand, harvesting every tree at the same date would result in excess demand for timber at the other dates and would not occur in equilibrium. Once prices are endogenized, a Hotelling-like result must re-emerge where an extractor is *indifferent* whether to harvest a tree immediately or later. If the value of the site is negligible, the value of the tree must grow at the rate of interest. If the tree is young and its wood volume is growing faster than the rate of interest, then the price of timber will be declining by enough that the young trees are harvested, although growing fast, before they decline further in value. When their uncut contemporaries become old and slow-growing, the price of their timber must rise by enough that the value of every tree continues to rise at the rate of interest. That is, the equilibrium price path can be $\cup$-shaped.[4]

An important result in the Hotelling literature is Herfindahl's insight that in a competitive equilibrium (or socially optimal allocation), reserves with a lower extraction cost are exhausted before reserves with a higher extraction cost are exploited. We show that similar results arise in the case of trees. If trees of differing vintages sit on sites of the same value, older trees would be harvested before younger trees. If trees of the same age sit on sites of different values, trees on the more valuable sites are harvested first. Because the Mitra-Wan-Salo-Tahvonen formulation with replanting is so general, we should expect to find similar Hotelling-like properties in some of its special cases.

We proceed as follows. In Section 2, the benchmark case is considered in

[4] In the Hotelling literature, $\cup$-shaped price paths can arise because of cost-reducing change in the extraction technology or because the exhaustible is a durable (Levhari and Pindyck, 1981).

which all trees are of the same vintage and sit on sites of the same value. In Section 3, we consider trees of the same vintage on sites of different values. In Section 4, we consider trees of different vintages on sites of the same value. Section 5 concludes by connecting our results to the more general model with replanting.

2 The Benchmark: Trees of the Same Vintage on Sites of the Same Value

To simplify at the outset, consider a set of trees of age a at initial date $t = 0$. Let $f(a + t)$ be the volume of wood available in each uncut tree at t (≥ 0). Assume that, following the cutting of a tree, its site is used for some other type of activity with exogenous value V (discounted to the time the tree is harvested). If there is a cost to cutting down the tree, V should be interpreted as the value of the site *net* of the cutting cost. Let $P(t)$ denote the price of timber per volume of wood at time t. Each tree owner is assumed to take the price path of timber as given. Denote as λ the present value of subsequently harvesting the tree at the optimal time and afterward using the site. Then, since the harvesting time is chosen optimally,

$$[P(t)f(a + t) + V]e^{-rt} \leq \lambda \text{ for } t \geq 0. \tag{1}$$

Suppose that all trees that become sources of timber are initially the same age. Suppose the demand for wood is stationary and depends only on the current price. That is, $D(P(t))$ is the cubic feet of wood per unit time that is demanded when the market price of timber is $P(t)$. Denote the rate at which trees of initial age a are felled at t as $q(t)$. Since each tree then contains $f(a+t)$ cubic units of wood, the *supply of timber* per unit time is $f(a+t)q(t)$. The market clears at t if and only if:

$$f(a+t)q(t) = D(P(t)). \tag{2}$$

For sufficiently high prices, demand may be choked off. In that event, equation (2) requires that $q(t) = 0$. If (1) holds as a strict inequality, the supply of timber equals zero. Any price that induces strictly positive demand for timber must solve the following equation:

$$P(t)f(a+t) + V = \lambda e^{rt} \text{ for all } t \text{ for which } D(P(t)) > 0. \tag{3}$$

If $V > 0$, the discounted value of cutting optimally will be strictly positive and each owner will want to harvest the trees eventually. This gives rise to the following endpoint condition:

$$\int_{t=0}^{\infty} q(t)dt = S(0), \tag{4}$$

where $S(0)$ denotes the exogenous number of trees at the outset.

For any λ, equation (3) can be solved for $P(t;\lambda)$ and the result can be substituted into equation (2) to obtain $q(t;\lambda)$.

$$q(t;\lambda) = \frac{D\left(\frac{\lambda e^{rt}-V}{f(a+t)}\right)}{f(a+t)}. \tag{5}$$

The notations $P(t;\lambda)$ and $q(t;\lambda)$ are intended to remind the reader that the functions depend on the multiplier λ, which is as yet undetermined. Now substitute (5) into equation (4) to *determine* the equilibrium value of λ :

$$\int_{t=0}^{\infty} \frac{D\left(\frac{\lambda e^{rt}-V}{f(a+t)}\right)}{f(a+t)} dt = S(0). \tag{6}$$

Since the left-hand side of equation (6) can be made to exceed or fall short of $S(0)$ by suitable choice of λ and since it is continuous in this variable, there exists one or more solutions to equation (6). Since the left-hand side of (6) is strictly decreasing in λ, however, the solution is unique. Denote it λ^*. To determine the equilibrium paths of extraction and price, set $q(t) = q(t;\lambda^*)$ and $P(t) = P(t;\lambda^*)$.

Since equation (3) holds over time, the shape of the equilibrium price path can be deduced by differentiating it:

$$f\dot{P} + P\dot{f} = rPf + rV, \tag{7}$$

where a single dot over a function denotes its first derivative and two dots would denote its second derivative. Hence,

$$\frac{\dot{P}(t)}{P(t)} = \left(r - \frac{\dot{f}(a+t)}{f(a+t)}\right) + \frac{rV}{f(a+t)P(t)} \text{ for } i = 1,\ldots,N. \tag{8}$$

Since the asset grows in volume, $\dot{f} > 0$. Moreover, we assume that $\ddot{f} < 0$; hence, the function $\dot{f}/f$ itself strictly decreases over time. The intuition underlying (8) is clear. The higher V is, the higher the capital gain must be to induce some owners to let their trees continue to grow. Similarly, the faster the percentage rate of growth in the volume of wood, the slower must be the increase in the price of timber to induce some owners to cut their trees down.

The equilibrium price path may well be $\cup$-shaped. To provide the simplest example, suppose that the site value is zero ($V = 0$). Assume that $\dot{f}(a)/f(a) > r$ but that as $t \rightarrow \infty$, $\dot{f}(a+t)/f(a+t) < r$. Then equation (8) implies that the price path is initially *downward-sloping*, unlike the case of a pure depletable, but eventually begins to rise. During the phase where the wood content of the tree grows faster than the rate of interest, the price of the tree must *strictly decline* so that the owner of the tree is indifferent about when to cut it; during the phase when the wood content of the tree increases by less than the rate of interest, the price of the tree must rise over time to preserve the owner's indifference about cutting times.

We now investigate the sensitivity of this equilibrium price path to changes in the exogenous parameters. Suppose the real rate of interest increased. Since condition (4) does not contain r, it must be satisfied with equality both before and after the interest rate change. It follows that the new path must cross the old one. Given differential equation (8), whenever the two paths cross, the path with the larger interest rate will be rising more rapidly and must cut the other path from below. Hence, while both paths may be $\cup$-shaped, they cannot cross more than once. It follows that an increase in the real rate of interest will induce the initial price to drop. Moreover, if there exists a finite choke price above which demand for timber is zero, then an increase in the rate of interest will cause the stock of trees to be exhausted more rapidly.

An exogenous increase in V will, for the same reasons, produce the same effect. Since the argument is unchanged, we will not repeat it.

An exogenous increase in the initial stock of trees will result in an equilibrium price path that is uniformly lower. Since the initial stock does not enter equation (8), both price paths must satisfy the identical differential equation; if they crossed at time t, they would have to coincide at all earlier and later times as well. It follows that either the paths coincide throughout or one lies uniformly above the other. In the latter case, demand will be uniformly larger on the lower price path. Hence, if the initial stock of trees is exogenously increased, the entire price path lies uniformly below the original path even if both are ∪-shaped; otherwise, excess supply would result.

2.1 Trees of the Same Vintage on Sites of Different Values

Suppose that the sites beneath some of the trees are especially valuable. Let $S_i(0)$ denote the number of trees on sites of value V_i for $i = 1, 2$ where $V_2 > V_1$. Then, by a trivial extension of the arguments surrounding equations (1) and (3), the market-equilibrium price must solve:

$$P(t) = \frac{1}{f(a+t)} \min\left(\lambda_1 e^{rt} - V_1, \lambda_2 e^{rt} - V_2\right), \tag{9}$$

where λ_i denotes the present value of cutting optimally a tree on a site worth V_i. Assuming that $V_i > 0,\ \lambda_i > V_i > 0$ for $i = 1, 2$.

In equilibrium, no tree on a site worth V_i will be cut at t if $P(t) <$

$(\lambda_i e^{rt} - V_i)/f(a+t)$. This implies that, in equilibrium, the multipliers $\{\lambda_i\}$ must have a particular relationship. For example, if $\lambda_1 \geq \lambda_2 > 0$, the first argument of the "min" function on the right-hand side of equation (9) would be strictly larger at *all* times and no tree on sites worth V_1 would *ever* be cut. This cannot occur in an equilibrium since it would not be optimal for the owners of trees on such sites. If, on the other hand, $0 < \lambda_1 < \lambda_2$, the second component will eventually exceed the first; indeed, it would exceed the first from the very outset if, in addition, $\lambda_1 - V_1 \leq \lambda_2 - V_2$. In this case, the second argument of the "min" function on the right-hand side of equation (9) would be strictly larger at *all* times and no tree on a site worth V_2 would *ever* be cut. This cannot occur in an equilibrium since it would not be optimal for the owners of trees on such sites. Hence, in any equilibrium in which trees on both types of sites are cut, the two multipliers must be set so that the following inequalities are satisfied:

$$0 < \lambda_1 < \lambda_2 \text{ but } \lambda_1 - V_1 > \lambda_2 - V_2. \tag{10}$$

In that case, the second component of the "min" function will be strictly smaller for $t \in [0, \hat{t})$ and the first component will be strictly smaller for $t \in [\hat{t}, \infty)$, where $\hat{t}$ is the unique root of the equation $g(\hat{t}) = 0$, and $g(t)$ is defined as follows:

$$g(t) = \lambda_2 e^{rt} - V_2 - (\lambda_1 e^{rt} - V_1) = (\lambda_2 - \lambda_1)e^{rt} + (V_1 - V_2). \tag{11}$$

It is straightforward to verify that there is a single root solving

$$g(\hat{t}) = 0. \tag{12}$$

To verify this, note that $g(t)$ is continuous in t and that $g(0) < 0$ but $g(t) > 0$ for sufficiently large t. This establishes the existence of *at least* one root. To establish that the root is unique, we verify by inspection of equation (11) that the function is strictly increasing. Since $g(t)$ is strictly increasing and has a root at $\hat{t}$, it must be strictly negative for $t \in [0, \hat{t})$ and strictly positive for $t \in [\hat{t}, \infty)$. It follows that

$$\frac{\lambda_1 e^{rt} - V_1}{f(a+t)} \geq \frac{\lambda_2 e^{rt} - V_2}{f(a+t)} \text{ for } t \leq \hat{t}. \tag{13}$$

Consequently, prior to $\hat{t}$, the market price is strictly below the first component of the "min" function and only the trees on the more valuable sites (worth V_2) are harvested. In the second phase, the market price is strictly below the second component of the "min" function and only the trees on the less valuable sites (worth V_1) are harvested.

The preceding arguments easily generalize. If there are initially sites with N distinct values ($V_N > \ldots > V_1$), then the equilibrium price must solve

$$P(t) = \frac{1}{f(a+t)} \min\left(\lambda_1 e^{rt} - V_1, \ldots, \lambda_N e^{rt} - V_N\right). \tag{14}$$

If trees on sites of every value are to be harvested eventually, then each component of this "min" function must be smallest over some time interval.

For this to occur, it is necessary to set $\{\lambda_i\}$ so that

$$\lambda_1 < \ldots < \lambda_N \text{ but } \lambda_1 - V_1 > \ldots > \lambda_N - V_N. \tag{15}$$

But this implies that trees on more valuable sites will be harvested in equilibrium before trees on less valuable sites.

For completeness, we return to the case where $N = 2$ and describe how the two multipliers would be determined. Let $q_i(t; \lambda_1, \lambda_2)$ denote the rate at which trees on sites worth V_i are cut for $i = 1, 2$. Then,

$$D(P(t)) = q_2(t; \lambda_1, \lambda_2) f(a + t) \text{ for } t \in [0, \hat{t}). \tag{16}$$

and

$$D(P(t)) = q_1(t; \lambda_1, \lambda_2) f(a + t) \text{ for } t \in [\hat{t}, \infty). \tag{17}$$

Since in equilibrium owners of trees on each type of site eventually harvest them, the following two endpoint conditions hold:

$$\int_{t=0}^{\hat{t}} \frac{D\left(\frac{\lambda_2 e^{rt} - V_2}{f(a+t)}\right)}{f(a + t)} dt = S_2(0) \tag{18}$$

and

$$\int_{t=\hat{t}}^{\infty} \frac{D\left(\frac{\lambda_1 e^{rt} - V_1}{f(a+t)}\right)}{f(a + t)} dt = S_1(0). \tag{19}$$

Equations (12), (18), and (19) determine the three unknowns—$\hat{t}, \lambda_1,$ and λ_2.

The shape of the equilibrium price path can be deduced by differentiating equation (14) to obtain:

$$\text{sign } \dot{P} = \text{ sign } \left(r - \frac{\dot{f}(a+t)}{f(a+t)} \right) + \frac{rV_i}{f(a+t)P(t)} \text{ for } i = 1, \ldots, N. \quad (20)$$

Since every tree on a higher valued site is extracted before any tree on a lower valued site and $V_{i+1} > V_i$, $P(t)$ is not differentiable across transitions from one site to the next. In particular, $P(t)$ is kinked at each transition with the left-derivative strictly larger than the right-derivative.

It is therefore possible for the equilibrium price of timber to have *two* phases of rising prices separated by a phase of falling prices. For example, assume $N = 2, V_1 = 0,$ and $V_2 > 0$. Suppose that $\lambda_1 e^{rt^*} = \lambda_2 e^{rt^*} - V_2$ so the transition occurs at t^*. Finally, assume that the following inequalities hold:

$$\frac{rV_2}{f(a+t^*)P(t^*)} > \frac{\dot{f}(a+t^*)}{f(a+t^*)} - r > 0. \quad (21)$$

This inequality ensures that immediately *after* trees on the worthless site (site 1) begin to be cut, the price of timber must fall, since trees are still young enough to be growing faster than the rate of interest. But immediately *before* that transition, the price of timber must be rising because—in the absence of such capital gains—the trees on the valuable site would all be harvested to gain access to the land underneath. Hence,

$$\frac{\dot{P}(t^*)^-}{P(t^*)} > 0 \text{ but } \frac{\dot{P}(t^*)^+}{P(t^*)} < 0.$$

Tree growth after t^* eventually falls below the rate of interest and the second phase of increasing prices begins. During this second phase, the growth of wood volume is so slow that a capital gain per unit volume is required to keep people from harvesting every tree.

3 Trees of Different Vintages on Sites of the Same Value

Suppose that initially there are trees of *two* different vintages. In particular, suppose initially there are $S_i(0)$ trees of age a_i, for $i = 1, 2$. Let the larger subscript refer to the older cohort of trees ($a_2 > a_1$) and assume older trees contain more wood: $f(a_2 + t) > f(a_1 + t)$. Assume that the site under every tree regardless of age is worth the same amount, V.

Then, by a trivial extension of the arguments surrounding equations (1) and (3) in the previous section, the market-equilibrium price must solve:

$$P(t) = \min\left(\frac{\lambda_1 e^{rt} - V}{f(a_1 + t)}, \frac{\lambda_2 e^{rt} - V}{f(a_2 + t)}\right), \tag{22}$$

where λ_i denotes the present value of cutting optimally a tree of initial age a_i. Assuming that $V > 0$, $\lambda_i > V > 0$ for $i = 1, 2$.

In equilibrium, no tree of initial age a_i will be cut at t if $P(t) < (\lambda_i e^{rt} - V)/f(a_i + t)$. This implies that, in equilibrium, the multipliers must have a particular relationship. For example, if $\lambda_1 \geq \lambda_2 > 0$, the first argument of the "min" function on the right-hand side of equation (22) would be strictly larger at *all* times and no tree of initial age a_1 would *ever* be cut. This cannot occur in an equilibrium since it would not be optimal for the owners of these younger trees. If, on the other hand, $0 < \lambda_1 < \lambda_2$, the second component will eventually exceed the first; indeed, it would exceed the first from the very outset if, in addition, $(\lambda_1 - V)/f(a_1) \leq (\lambda_2 - V)/f(a_2)$. In this case, the second argument of the "min" function on the right-hand side of equation (22) would be strictly larger at *all* times and no tree of initial age a_2 would *ever* be cut. This cannot occur in an equilibrium since it would not be optimal for the owners of these older trees. Hence, in any equilibrium in which both vintages of trees are cut, the two multipliers must be set so that the following inequalities are satisfied:

$$0 < \lambda_1 < \lambda_2 \text{ but } \frac{\lambda_1 - V}{f(a_1)} > \frac{\lambda_2 - V}{f(a_2)}. \tag{23}$$

In that case, the second component of the "min" function will be strictly smaller for $t \in [0, t^*)$ and the first component will be strictly smaller for $t \in [t^*, \infty)$, where t^* is the unique root of the equation $h(t^*) = 0$, and $h(t)$ is defined as follows:

$$h(t) = \frac{f(a_1 + t)}{\lambda_1 e^{rt} - V} - \frac{f(a_2 + t)}{\lambda_2 e^{rt} - V}. \tag{24}$$

It is straightforward to verify that there is a single root solving

$$h(t^*) = 0. \tag{25}$$

To verify this, note that $h(t)$ is continuous in t and that $h(0) < 0$ but $h(t) > 0$ for sufficiently large t. This establishes the existence of *at least* one root. To establish that the root is unique, we verify that at every root, $\dot{h}(t^*) > 0$.

Hence, there can be only one root. To verify this, differentiate equation (24):

$$\dot{h}(t) = \left[\frac{\dot{f}(a_1+t)}{\lambda_1 e^{rt} - V} - \frac{\dot{f}(a_2+t)}{\lambda_2 e^{rt} - V}\right] + r\lambda_1 e^{rt}\left[\frac{\lambda_2}{\lambda_1}\frac{f(a_2+t)}{(\lambda_2 e^{rt} - V)} - \frac{f(a_1+t)}{\lambda_1 e^{rt} - V}\right] \tag{26}$$

Since in equilibrium $\lambda_2/\lambda_1 > 1$ and equation (25) holds, the second term in the square brackets of (26) is strictly positive at $t = t^*$. As for the first term in square brackets, it is the difference of two positive fractions, the first of which is larger than the second since it has a larger numerator and a smaller denominator. Hence, the second term in square brackets is also strictly positive and $\dot{h}(t^*) > 0$.

Since $h(\cdot)$ has a unique root and is strictly increasing at that root, the function must be strictly negative for $t \in [0, t^*)$ and strictly positive for $t \in [t^*, \infty)$. It follows that

$$\frac{\lambda_1 e^{rt} - V}{f(a_1+t)} \geq \frac{\lambda_2 e^{rt} - V}{f(a_2+t)} \text{ for } t \leq t^*. \tag{27}$$

Consequently, prior to t^*, the market price is strictly below the first component of the "min" function and only the older trees (of initial age a_2) are harvested. In the second phase, the market price is strictly below the second component of the "min" function and only the *younger* trees (of initial age a_1) are harvested.

The preceding arguments easily generalize. If there are initially N vintages of trees,

$$P(t) = \min\left(\frac{\lambda_1 e^{rt} - V}{f(a_1+t)}, \ldots, \frac{\lambda_N e^{rt} - V}{f(a_N+t)}\right). \tag{28}$$

If every vintage is to be harvested eventually, then each component of this "min" function must be smallest over some interval. For this to occur, it is necessary to set $\{\lambda_i\}$ so that

$$\lambda_1 < \ldots < \lambda_N \text{ but } \frac{\lambda_1 - V}{f(a_1)} > \ldots > \frac{\lambda_N - V}{f(a_N)}. \tag{29}$$

But this implies that older trees will be harvested in equilibrium before younger trees.

The shape of the equilibrium price path can be deduced by differentiating equation (28) to obtain:

$$\text{sign } \dot{P} = \text{ sign } \left(r - \frac{\dot{f}(a_i + t)}{f(a_i + t)} \right) + \frac{rV}{f(a_i + t)P(t)} \text{ for } i = 1, \ldots, N. \tag{30}$$

Since every older tree is extracted before any younger tree and $a_{i+1} > a_i$, $P(t)$ is not differentiable across vintage transitions. In particular, $P(t)$ is again kinked at each transition with the left-derivative larger than the right-derivative; this property must arise since $P(t)$ is the lower envelope of the N differentiable functions $(\frac{\lambda_i e^{rt} - V}{f(a_i+t)})$.

It is, however, possible in this case for the price of timber to have two rising phases, each preceded by a phase where the timber price falls. As a result, the path of timber prices can have the shape of a script W, which is smooth at its two local minima but kinked at its local maximum.

Consider the case where $N = 2$ and $V = 0$. Assume

$$\frac{\dot{f}(a_2)}{f(a_2)} > r. \tag{31}$$

Suppose that $\frac{\lambda_1 e^{rt^*}}{f(a_1+t^*)} = \frac{\lambda_2 e^{rt^*}}{f(a_2+t^*)}$ so the transition occurs at t^*. Finally, assume that the following inequalities hold:

$$\frac{\dot{f}(a_1+t^*)}{f(a_1+t^*)} > r > \frac{\dot{f}(a_2+t^*)}{f(a_2+t^*)}. \tag{32}$$

Inequality (31) ensures that the older trees are initially growing faster than the rate of interest. In order for the owner of such trees to harvest any of them and invest the proceeds at the smaller rate of interest, the price of timber must be declining. However, inequality (32) ensures that by the time of the transition (t^*), the growth rate of the older trees will have declined below the rate of interest while the younger trees are still growing faster than the rate of interest. As a result, just before the transition occurs, the price of timber will be rising; but just afterward, it must be falling. Eventually, of course, these younger trees age and their growth rate declines below the rate of interest. Hence, the path of the timber price has the shape of a script W.

Consider the following concrete example. Let $V = 0, r = 5\%$ and $f(a_i + t) = (a_i + t)^\eta$. Then $\dot{f}(a_i + t)/f(a_i + t) = \eta/(a_i + t)$. If $\eta = 1/2$, then the rate of growth of the trees declines monotonically to zero and reaches 5% when trees are 10 years old. Assume that initially trees are either one year old ($a_1 = 1$) or five years old ($a_2 = 5$). Then the growth rate on the older trees declines to 5% when $t_{min2} = 5$, and the growth rate of the younger trees declines to 5% when $t_{min1} = 9$. At each date, trees of the particular vintage will be 10 years old. Assign $\lambda_1 = 1$ and $\lambda_2 = e^{.2} > \lambda_1$. This assignment of multipliers ensures that when each curve, $\lambda_i e^{rt}/(t + a_i)^{1/2}$ (for $i = 1, 2$), reaches its respective minimum ($t_{\min 1} = 9, t_{\min 2} = 5$), the two functions have the same value.[5] Note finally that $1 < e^{.2}/5^{1/2}$, so $\lambda_1/f(a_1) < \lambda_2/f(a_2)$ so condition (29) is satisfied. In this example, the lower envelope of the two functions traces out a script W.

4 Conclusion

This paper contributes to the literature that endogenizes the price (or implicit price) of timber in forestry models. By making the unrealistic simplification that replanting is infeasible, we are able to connect the literature on tree cutting with the literature on nonrenewable, nongrowing resources launched by Hotelling (1931), developed by Herfindahl (1967), and surveyed recently by Gaudet (2007). The rest of the forestry literature has considered the more challenging problem in which replanting occurs every time a tree is cut. The formulation of this model is extremely general. Trees can be any age, $s = 1, \ldots n$ (for any integer n), and lose all economic value if cut when older. The initial distribution of vintages is arbitrary. Older trees contain weakly more wood. The "indirect approach" is obviously the sensible way to approach such a formidable problem despite the concerns of Stokey et al. (1989) mentioned in the introduction. The focus in this literature has not yet been turned to the path of the marginal utility of timber consumption (which would equal the equilibrium timber price in a competitive model). Nonetheless, it would be surprising if some of the properties of our simplified model do not emerge in models with replanting. Suppose, for example, that

[5] Given this particular assignment of multipliers, one can compute the strictly positive cumulative demand for trees of each vintage along the price path. This will be the equilibrium price path if the initial stocks of trees in this example match the induced demand for them.

the planner starts with every tree 100 years old in a world where trees do not grow at all until they reach that age, and then grow at first faster and then slower than the rate of interest, dying on their 200th birthday. Clearly the implicit price path of timber must at first resemble that of our case where the exogenous site value is zero and all trees have the same vintage. Some trees will be cut and others will continue to grow. For this to occur in a competitive equilibrium, the extractor must be indifferent whether he cuts immediately or later, just as in the Hotelling model. Moreover, since trees of the initial age grow much faster than the rate of interest and the site value is so small, no one would be willing to cut them immediately unless their price was falling. It is left to future work to map out which properties of the price path and allocation in our model where replanting is infeasible carry over to the model with replanting.

5 References

Amacher, Gregory, Markku Ollikainen and Erkki Koskela, "Economics of Forest Resources," The MIT Press, Cambridge, 2009.

Faustmann, M., "Calculation of the Value which Forest Land and Immature Stands Possess, 1849 paper reprinted in *Journal of Forest Economics* vol. 1, p. 7-44, 1995.

Gaudet, Gérard, "Natural Resource Economics under the Rule of Hotelling,"' *Canadian Journal of Economics*, vol. 40, p. 1033-59, 2007.

Gray, L.C., "Rent under the Assumptions of Exhaustibilility," *Quarterly Journal of Economics*, vol. 28, p. 466-89, 1914.

Herfindahl, O. C., "Depletion and Economic Theory," in *Extractive Resources and Taxation*, edited by M. Gaffney, University of Wisconsin Press, Madison, 1967.

Hotelling, H., "The Economics of Exhaustible Resources," *Journal of Political Economy*, vol. 30, p. 131-75, 1931.

Levhari, David and Robert S. Pindyck, "The Pricing of Durable Exhaustible Resources," *Quarterly Journal of Economics*, vol. 96, p. 365-77, 1981.

Lyon, Kenneth, "Mining of the Forest and the Time Path of the Price of Timber," *Journal of Environmental Economics and Management*, vol. 8, p. 330-44, 1981.

Lyon, Kenneth and Roger Sedjo, "An Optimal Control Theory Model to Estimate the Regional Long-Term Supply of Timber," *Forest Science*, vol. 29, 798-812, 1983.

Mitra, Tapan and Henry Wan, "Some Theoretical Results on the Economics of Forestry" *Review of Economic Studies*, vol. 52, p. 263-82, 1985.

Salo, Seppo and Olli Tahvonen, "On Equilibrium Cycles and Normal Forests in Optimal Harvesting of Tree Vintages," *Journal of Environmental Economics and Management*, vol. 44, p. 1-22, 2002.

Salo, Seppo and Olli Tahvonen, "On the Economics of Forest Vintages," *Journal of Economic Dynamics and Control*, vol. 27, p. 1411-35, 2003.

Samuelson, Paul "Economics of Forestry in an Evolving Society," *Economic Inquiry*, vol. 14, p. 466-92, 1976.

Sedjo, Roger and Kenneth Lyon, *The Long-Term Adequacy of World Timber Supply*, Resources for the Future, Washington, D.C., 1990.

Stokey, Nancy and Robert Lucas with Ed Prescott, "Recursive Methods in Economic Dynamics," Harvard University Press, Cambridge, 1989.