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DISCUSSION PAPER

Optimal Control of Spatial-Dynamic Processes

The Case of Biological Invasions

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Optimal Control of Spatial-Dynamic Processes: The Case of Biological Invasions

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Abstract

This study examines the spatial nature of optimal bioinvasion control. We develop a spatially explicit two-dimensional model of species spread that allows for differential control across space and time, and we solve for optimal spatial-dynamic control strategies. We find that the optimal strategies depend in interesting ways on the shape of the landscape and the location, shape, and contiguity of the invasion. For example, changing the shape of the invasion or using landscape features to reduce the extent of exposed invasion edge can be an optimal strategy because it reduces long-term containment costs. We also show that strategies should be targeted to slow or prevent the spread of an invasion in the direction of greatest potential long-term damages. These spatially explicit characterizations of optimal policies contribute to the largely nonspatial literature on controlling invasions and our general understanding of how to control spatial-dynamic processes.

Key Words: invasive species, spatial-dynamic processes, spatial spread, reaction-diffusion, management, cellular automaton, eradication, containment, spatial control, integer programming

JEL Classification Numbers: Q16, Q20, Q22, Q23, Q54, Q57

Contents

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Rebecca S. Epanchin-Niell and James E. Wilen*

1. Introduction

Much of the economic research on bioinvasion management has framed the issue as a pest-control problem, in which the population density of the invader is controlled. This literature generally has focused on the aggregate pest population, without considering its spatial characteristics.[1] However, a critical feature of bioinvasions is that they unfold over time and space. They generally begin with the arrival of just one or a few individuals to a region. The population density of the invader then may increase at the arrival site by reproduction. In addition, the population may spread over space by dispersal, so that the initial population of invaders eventually can impact locations far from the site of colonization. Bioinvasions are thus driven by spatial-dynamic processes, rather than by simpler dynamic processes.

Although existing analytical work generally has abstracted away from the spatial features of invasions, it has provided important insights on how optimal invasion management depends on costs, damages, discount rate, growth and spread dynamics, and even the size of the invasion and landscape. Most of these findings have been derived using spatially implicit models of invasion spread that yield conclusions about *when* and *how much* to control.[2] Since few studies have explicitly considered the spatial characteristics of bioinvasions, there is less understanding about *where* to optimally allocate control efforts or the effect of spatial landscape and invasion characteristics on optimal control choices.

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[1] See, for example, Pannell 1990; Deen et al.1993; and Saphores (2000).

[2] See, for example, Eiswerth and Johnson (2002); Olson and Roy (2002, 2008); and others reviewed by Epanchin-Niell and Hastings (2010) and Olson (2006).With spatially implicit models, the state variable generally measures the size or extent of invasion over an entire landscape without respect to space. The state equation is often an ordinary differential equation that describes how the size or extent of the invasion varies over time as a function of itself and the quantity of control. This approach casts invasion control as a dynamic problem whose solution is a time path for the optimal control policy.

This paper develops a general bioeconomic model of bioinvasions that incorporates a spatial-dynamic spread process and allows us to explicitly characterize various aspects of space. The model departs from approaches based on optimal control and dynamic programming by casting bioinvasions in a manner that allows us to for optimal policies using an efficient integer-programming algorithm. As a result, we can handle large-dimension characterizations of explicit space with realistic features, without being hindered by the curse of dimensionality.

We use the model to examine optimal policies over a range of bioeconomic parameters, spatial configurations, and initial invasion types. The more interesting results show how the initial invasion's shape, contiguity, and location in the landscape influence the qualitative characteristics of optimal control policies. We uncover a range of optimal solutions, from immediately eradicating the invasion to abandoning control, and from slowing and then stopping an invasion to slowing and then abandoning control. We provide intuition about why various sequences of controls are optimal, focusing on how topology of the invasion influences border length and containment costs. We also show how optimal policies utilize landscape features, including range borders, as part of their solutions. Our results provide important new insights into how adding space to dynamic problems matters and how spatial-dynamic solutions differ from simpler dynamic solutions.

2. Related Literature

Understanding spatial aspects of invasion management requires a spatially explicit model of invasion spread. In its most general form, a spatial spread process may be characterized with a partial differential equation (PDE) over continuous time and space. Because solving PDEs is notoriously difficult, however, characterizing all but the simplest forms of a spatial-dynamic spread process analytically is not possible. Solving a problem that involves optimization of a system characterized by a PDE is even more difficult. There is a paucity of literature in economics, mathematics, and optimization theory on characteristics of optimally controlled partial differential state equation systems.

Several recent studies attempt to uncover general properties of spatial-dynamic systems using analytical approaches. The most general is the elegant work by Brock and Xepapadeas (2004, 2008), who derive modified Pontryagin conditions for the optimal control of a system governed by PDEs. Brock and Xepapadeas apply their method to the problem of optimally harvesting from a fishery with density-dependent growth and random local dispersal. They focus on the steady state of the system—in particular, whether the steady state is uniform over space or exhibits inhomogeneities. They simplify the spatial part of the system by assuming that space is a one-dimensional circle, keeping the dimensionality to minimal size, and avoiding more complex boundary conditions.

In parallel work, Sanchirico and Wilen (1999, 2005, 2007) model a renewable resource system in continuous time and discrete space, which allows the general PDE system to be converted into a system of ordinary differential equations of dimension equal to the number of patches. Sanchirico and Wilen (2005) apply optimal control methods to the system to show how the equilibrium configuration of biomass and harvest depend on cost, price, and discount parameters, as well as on the dispersal system that determines the links between patches. Sanchirico et al. (2010a) complete the modeling of optimal metapopulation harvesting by solving for the approach path to equilibrium using numerical methods. Costello and Polasky (2008) characterize the fisheries system as discrete in time and space, allowing them to use dynamic programming to analytically characterize features of the equilibrium of a metapopulation fishery system in the presence of stochasticity.

This recent economic literature on optimal renewable resource management is not directly applicable to characterizing optimal bioinvasion management for a variety of reasons. First, solutions to fisheries problems rarely involve permanent and complete closure of the fishery or system-wide extinction. Instead, they define an infinite stream of harvest rates across space. In contrast, while relevant control options for bioinvasions include policies that define an infinite stream of controls across space (e.g., applying controls to stop an invader’s advance), they also include extremes such as system-wide eradication or complete abandonment of control efforts. Including eradication seems critical for a realistic depiction of bioinvasion control, but it complicates the solution since eradication policies eliminate damages and control options in finite time. As a result, characterizing optimal bioinvasion control solutions requires comparing finite and infinite horizon solutions over the parameter space (Wilen 2007).

Second, while the steady-state equilibrium is arguably the more interesting part of renewable resource problems, the approach path is more important for bioinvasion problems. Solving for approach paths in dynamic analysis is always difficult and generally requires numerical methods. Numerical methods, however, face an enhanced curse of dimensionality for spatial-dynamic problems, limiting the size and complexity of the spatial landscape that can be analyzed.

Third, aside from the extra dimensions associated with incorporating space, it is desirable to characterize space in realistic ways. For example, how do different shapes of the initial invasion and the landscape influence optimal policies? How does heterogeneity of costs or damages over space affect policies? These kinds of questions are relevant to real-world bioinvasion control but generally are beyond the scope of analytical modeling.

Fourth, the end-point conditions for spatial-dynamic problems only recently have been articulated (Brock and Xepapadeas 2008) and are difficult to incorporate in numerical solutions for all but the simplest of spatial structures.

To address some of these difficulties, recent research has taken a variety of approaches. Some studies employ heuristic approaches that compare a predefined set of control strategies by simulating their outcomes, rather than finding a single optimal policy (e.g., Grevstad 2005; Higgins et al. 2000; Wadsworth et al. 2000). A study by Hof (1998) assumed the objective of minimizing the population of a spreading pest, subject to a budget constraint, exponential population growth, and linear costs. These assumptions simplify the problem but also mandate maximal control in all time periods. Other studies simplify by solving only for the equilibrium optimality conditions, rather than the path by which that equilibrium is optimally achieved (Albers et al. 2010; Potapov and Lewis 2008; Sanchirico et al. 2010b). Some approaches incorporate realistic features of control options and the characterization of space but make the problem tractable by reducing the dimension. For example, Albers et al. (2010) and Sanchirico et al. (2010b) identify equilibrium control solutions for a complex model of invasion spread and reproduction dynamics in a two-patch model allowing for multiple types of control.

Blackwood et al. (2010) take the closest to our approach by accounting for large-scale explicit space, but solve for optimal spatial-dynamic control policies over a finite horizon with no transversality condition. They model a pest that exhibits exponential growth over time and quadratic control costs, assumptions that allow them to cast the problem as a linear-quadratic control problem. Their assumptions of exponential growth allow damages at any location to become infinite in the absence of control and thus force all optimal management policies to drive the invasion population near zero. In addition, their study assumes that the invasive species causes no damages after the final time period (generally 10 years) and allows for both negative invasion and control levels.

This existing work has contributed three particularly important insights to the characteristics of optimal spatial invasion management: that it can be optimal to apply control heterogeneously across space; dispersal of the invader matters; and optimal control at each site depends not only on conditions there, but also at other sites.

3. A General Spatial-Dynamic Model of Bioinvasions

Despite these insights, the general spatially explicit bioinvasion optimization problem has not been characterized in a manner that allows for a range of control outcomes, captures explicit spatial features of the invasion and the landscape in which it occurs, and can be solved for a large landscape. Our approach accomplishes these by exploiting a model that is stripped to its essential features but retains important flexibility for incorporating explicit spatial processes.

We develop a spatially explicit, deterministic, discrete space-time model that allows for growth and spread of a species and differential control over time and space. In this problem, we focus on the situation in which an invasion has arrived, established itself, and been discovered within the focal landscape. Upon discovery, the initial invasion has some arbitrary character (e.g., size, shape, location) that may depend on the mechanism (e.g., animals, wind) responsible for its introduction. We then address the question of how this general, arbitrarily shaped, one-time invasion should be managed beginning when it is discovered to minimize the total costs and damages incurred from the invasion.

Although invasion spread can follow a variety of processes, we focus on species that exhibit random local dispersal, such that the spread can be represented as a reaction-diffusion process and approximated as a linear rate of spread.[3] We focus on this spread process because it is tractable and much can be learned from even this simplest case. We discuss how our results and conclusions generalize to alternative spread processes, including those that account for stochastic, rare, long-distance dispersal events and repeat invasions.

We model invasion spread in a two-dimensional landscape and allow for distinct landscape boundaries to represent the potential range of contiguous spread of an invasion. We model this landscape as a lattice or grid of cells that are linked by dispersal. Cells are either invaded or uninvaded, and the invasion spreads from invaded to adjacent uninvaded cells in each time period, approximating a constant rate of radial spread. In the absence of control efforts, the invasion spreads to fill the entire focal landscape.

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[3] Mathematically, reaction-diffusion equations describe how the concentration of one or more substances distributed in space changes under the influence of two processes: local reactions, in which the quantity of the substances can change, and diffusion, which causes the substances to spread out in space. When modeling the spread of biological invasions or other organisms, the reaction process characterizes the species population growth. Our reaction-diffusion assumption thus depicts the combined processes of reproduction and dispersal.

Our model incorporates two types of invasion control: clearing of invaded cells and preventing spread between invaded and uninvaded cells. Each control action has an associated cost, and combinations of control actions can be used to eradicate, contain, or slow the spread of the invasion.

The simplicity of this discrete spatial spread model and the specification of control options provide important benefits. First, we are able to depict the objective function and state equations in a manner that does not rely on dynamic programming or hill-climbing algorithms. Instead, our framework enables fast solutions to the spatial-dynamic optimal control problem—despite its high dimensionality—using integer programming.[4] In more general spatial-dynamic problems, high dimensionality limits the size of the problem that can be solved, thus limiting the questions that can be addressed (Konoshima et al. 2008; Potapov and Lewis 2008). A second benefit of the model's simplicity is that the intuition is more readily apparent from results despite the inability to derive an analytical solution to this complex problem. In the results presented here, we exploit the speed of the algorithm by solving hundreds of optimizations for a large span of the parameter space. This large array of results helps identify specific parametric assumptions and initial conditions that influence control strategies, which enables us to synthesize the intuition behind results.

Our most interesting findings demonstrate the manner in which boundaries and geometry of the invasion and the landscape matter. For example, the location of an invasion relative to the boundaries of its potential range affects the prospective damages and the control costs, thereby affecting optimal control policies and the net present value of the invasion. In addition, small

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[4] Even with binary invasion states (i.e., each cell in the landscape grid is either invaded or uninvaded in each time period), the size of this optimization problem grows exponentially with the size of the landscape. For example, there are 2^{25}, or more than 33 million potential configurations of invaded and uninvaded cells in a simple 5x5 landscape grid in a single time period, and hence simple search or enumeration algorithms cannot be used to solve this optimization problem with multiple time periods. The state of the art in computational algorithms for solving our kind of problem via conventional dynamic programming–based algorithms is summarized in Farias et al. (forthcoming). Their approximation technique solves a complex game theoretic equilibrium with 50 firms and 20 states per firm. This is approximate to a problem with a 13x13 grid. Our approach enables us to solve for optimal control of invasion spread in very large landscapes (greater than 25x25) and many time periods with integer-programming software using a branch and bound algorithm. Two features of our model setup facilitate its solution. First, we are able to specify the nonlinear invasion and control dynamics as sets of linear inequalities, which greatly facilitates the solution to integer-programming programs. Secondly, our spatial-dynamic model retains some of the useful properties of more general dynamic problems. Specifically, the ordered structure imposed by time (i.e., a topological sort) maintains the applicability of the Bellman principle of optimality, thus allowing the problem to be reduced to collection of simpler, overlapping subproblems.

differences in the shape and contiguity of the initial invasion can change the qualitative nature of optimal controls. This is often because the extent of the exposed invasion edge determines long-term containment costs. We find that employing landscape features and/or altering the shape of the initial invasion through spread or removal can optimally reduce the length of the exposed invasion front. We also show how optimal policies exhibit classic forward-looking behavior that characterizes optimal dynamic problems. In spatial-dynamic problems, however, optimal policies not only anticipate impacts over time, but they also look forward over space to determine where and when to apply various controls. In general, invasion control is targeted to slow the spread of an invasion in the direction of greatest potential local or long-term damages, or where the costs of achieving control are low.

In the next section, we describe the bioinvasion-spread model, economic model, and solution approach. In section 5, we describe the results and derive features of optimal policies for a number of specific spatial-dynamic problems, with an eye toward deriving general qualitative properties of these systems. In section 6, we summarize and discuss results in the context of existing invasion literature. We conclude in section 7 by highlighting some general principles about controlling spatial-dynamic processes that we have deduced from our bioinvasion case study.

4. Methods

4.1 Spread Model

We define the landscape as the total contiguous two-dimensional range over which the invader can spread, such that landscape boundaries represent ecological or physical barriers to spread. For example, aquatic species are confined to water bodies and may face particular temperature and depth requirements; many insect species are confined by the range of their hosts (e.g., wood-boring insects in forests); and, more generally, the potential range or spread of an invading species is bounded by physical or ecological barriers (e.g., temperature, elevation, soils).[5] Although these often are not hard boundaries but gradients of habitability, we capture these important constraints to spread by defining a finite landscape whose edges represent these

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[5] The potential range of an invading species generally can be predicted using ecological niche modeling (Elith et al. 2006; Peterson 2003).

boundaries. The landscape can take any shape to represent features of the potential invasion range, although we work primarily with rectangular, smooth shapes.

As is common in ecological models, such as metapopulation and other spatial approaches, we make space discrete. One can think of this either as a representation of a landscape consisting of discrete cells linked by dispersal or simply as an arbitrary means for dividing space into indexable units. One could arbitrarily divide the landscape into a very large number of cells; as the number approaches infinity, discrete space becomes a better and better representation of continuous space.

We model each cell as invaded or uninvaded rather than accounting for species density. This approach, which has been commonly employed in metapopulation models (e.g., Hanski 1999; Kawasaki et al. 2006), abstracts away from detailed population dynamics but still captures species' constraints to growth.

Invasion spread occurs through the combination of reproduction and dispersal. Many species spread primarily based on random, local movements. In this case, invasion spread can be approximated by a reaction-diffusion model, which predicts a linear rate of spread (Shigesada and Kawasaki 1997). Researchers have used this model to predict spread for many invasions and help inform management decisions (e.g., Sharov and Liebhold 1998). We employ a discrete space-time version of this spread pattern by assuming that the invasion spreads radially by one unit of space in each time period. Others also have employed this basic contact process for spread.[6]

Many invasions also spread by rare long-distance dispersal events (or stratified-diffusion processes), often driven by human assistance (Epanchin-Niell and Hastings 2010; Shigesada and Kawasaki 1997). Accounting for this type of dispersal and the potential for reinvasion from outside propagule pressure may be important for determining management of specific species. However, we focus on the simplest case, which provides intuition for the more general case and offers tractability.

More precisely, we represent the landscape as a grid of square cells that comprises the total potential extent of contiguous invasion. Each cell is labeled by its row *i* and column *j* in the

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[6] For example, Kawasaki et al. (2006) model invasion spread using discrete space in which landscape cells are invaded or uninvaded. They employ a probabilistic model in which spread occurs to adjacent cells. Here we focus on the deterministic model for tractability within an optimization framework.

landscape grid, and each cell can take one of two states: invaded ($x_{i,j} = 1$) or uninvaded ($x_{i,j} = 0$). In the absence of any human intervention, the species spreads from invaded cells to adjacent, uninvaded cells in each time period, based on rook contiguity. Thus, if cell (i,j) were invaded at time t, cells (i,j), (i,j+1), (i,j-1), (i+1,j), and (i-1,j) would be invaded in the next time period. In each subsequent time step, all cells sharing a contiguous border with an invaded cell also become invaded. When making space and time discrete, an appropriate combination of cell size and time unit must be selected to model specific species; these choices are closely linked because the model assumes that the invasion spreads into adjacent uninvaded space at a rate of one grid cell per unit of time.

4.2 Economic Model

We assume that the invasive species causes damages proportional to the area invaded. Hence, landscape-level damages at each point in time are directly proportional to the number of invaded grid cells, with marginal (and average) damages equaling d per cell invaded. Our model incorporates two types of control: preventing invader establishment in uninvaded cells and clearing invaded cells. We assume that the cost of preventing establishment of the invasion in a cell depends linearly on the propagule pressure from adjacent invaded cells. Thus, the cost of excluding invasion from a cell increases with the number of adjacent (rook contiguous) invaded cells and equals *invaded_neighbors*b*, where b is the cost of preventing invasion along each boundary and *invaded_neighbors* is the number of invaded adjacent cells (0 ≤ *invaded_neighbors* ≤ 4). Once a cell has been invaded, it remains so unless the invasion is removed from the cell at a cost e. The cost of clearing thus depends linearly on the area cleared. For a cleared cell to remain uninvaded in the following time periods, control must be applied to prevent reinvasion at a cost *invaded_neighbors*b*. If the entire landscape has been cleared, there are no subsequent control costs.

To parameterize this model, damages and costs are tied to the size of the cell and scaled accordingly. Similarly, the discount rate is scaled to match the unit of time. By separately parameterizing removal costs e and spread prevention costs b, this model allows flexibility in specifying control costs based on species characteristics. For many species, such as plants with long-lived seed banks or species exhibiting Allee effects, preventing establishment in an area may be much less costly than removing an established invasion from that area; this can be reflected in the choice of cost parameters. Alternatively, parameters e and b can be thought to represent the cost of clearing high-density versus low-density cells since newly invaded cells are likely to have lower invader density than fully established ones.

4.3 Optimization Set-Up and Solution Approach

Optimal control of the invasion requires minimizing the present value of the sum of control costs and invasion damages across space and time. With careful specification and choice of variables, we are able to formulate this nonlinear invasion problem as a set of linear constraints and objective function that can be solved as a binary integer-programming problem. We specify the problem as follows:

$$\text{minimize} \sum_{t\in T, t>0} \beta_t * \left(\sum_{(i,j)\in C} x_{i,j,t} d + \sum_{(i,j)\in C} y_{i,j,t} e + \sum_{(i,j,k,l)\in N} z_{i,j,k,l,t} b \right) \qquad (1)$$

subject to

$$x_{i,j,0} = \underline{x}_{i,j} \qquad \forall (i,j) \in C; \qquad (2)$$

$$y_{i,j,0} = 0 \qquad \forall (i,j) \in C; \qquad (3)$$

$$z_{i,j,k,l,0} = 0 \qquad \forall (i,j,k,l) \in N; \qquad (4)$$

$$x_{i,j,t} \geq x_{i,j,t-1} - y_{i,j,t} \qquad \forall (i,j) \in C, t \in T, t \geq 1; \qquad (5)$$

$$x_{i,j,t} \geq x_{k,l,t-1} - z_{i,j,k,l,t} - y_{i,j,t} \qquad \forall (i,j,k,l) \in N, t \in T, t \geq 1; \text{ and} \qquad (6)$$

$$x_{i,j,t} \in \{0,1\} \qquad \forall (i,j) \in C, t \in T; \qquad (7)$$

where the following is true:

$(i, j) \in C$ indexes cells by row *i* and column *j*, and *C* is the set of all cells in the landscape;

$(i, j, k, l) \in N$ indexes pairs of neighboring cells, where $(i, j) \in C$ is the reference cell, $(k, l) \in C$ is one of its neighbors, and *N* is the set of all neighboring cell pairs;

$t \in T$ indexes time, where $T = \{0,1,2,\ldots,T_{\max}\}$;

$x_{i,j,t} \in \{0,1\}$ is the state of cell (i, j) at time *t*, where $x_{i,j,t} = 1$ if the cell is invaded and $x_{i,j,t} = 0$ otherwise;

$y_{i,j,t} \in \{0,1\}$ is a binary-choice variable indicating if invasion is removed from cell (i, j) at time *t*, where $y_{i,j,t} = 1$ if the cell is cleared and $y_{i,j,t} = 0$ otherwise;

$z_{i,j,k,l,t} \in \{0,1\}$ is a binary-choice variable indicating if control efforts are applied along the border between cell (i, j) and cell (k,l) at time *t* to prevent spread from cell (k,l) to cell (i, j), where $z_{i,j,k,l,t} = 1$ if the border is controlled and $z_{i,j,k,l,t} = 0$ otherwise;

$\underline{x}_{i,j} \in \{0,1\}$ is the initial state (*t*=0) of invasion for cell (i, j);

β_t is the discount factor at time *t* (*t*>0), where $\beta_t = (1+r)^{1-t}$ and *r* is the discount rate;

d is the damage incurred per time period for each cell that is invaded;

e is the cost of removing invasion from a cell; and

b is the cost per time period of preventing invasion from a neighboring cell.

Equation (2) establishes the initial state of the landscape by defining which cells are invaded at *t*=0. Equations (3) and (4) specify that control efforts do not begin until the first time period. Condition (5) requires that a cell invaded in the previous time period remains invaded in the current time period unless removal efforts are applied. Equation (6) requires that cell (*i*,*j*) become invaded at time *t* if it had an invaded neighbor in the previous time period, unless invasion is removed from cell (i, j) or control is applied to prevent invasion from the invaded neighbor; this condition must hold for cell (i, j) with each of its neighbors. Individually, constraints (5) and (6) provide necessary conditions for a cell to be uninvaded at time *t*; together, constraints (5) and (6) provide sufficient conditions for a cell to be uninvaded at time *t*.

Specifically, an uninvaded cell (i, j) will become invaded at time t+1 unless spread is prevented at time t+1 from each adjacent invaded cell at time t or removal efforts are applied to cell (i, j) at time t+1. An invaded cell (i, j) at time t remains invaded at time t+1 unless removal efforts are applied to it at time t+1.

In this paper, we are interested in the optimal control solution over an infinite time horizon, but solving for this directly as an integer-programming problem would require specifying an infinite number of constraints and variables, making the problem infeasible. Instead, we solve for the infinite time horizon solution in a finite time horizon framework by taking advantage of Bellman's principle of optimality and specifying appropriate terminal values. With a sufficiently long time horizon, our system eventually reaches a steady-state equilibrium in which none to all of the landscape is invaded. In fact, with an infinite time horizon, time consistency requires that the system has reached this equilibrium if the invasion landscape remains unchanged between two time periods. In contrast, for a finite time horizon, the system can reach and maintain the steady-state equilibrium for many time periods but can depart from the steady state toward the end of the finite time horizon. To deal with this difficulty, the steady-state equilibrium solution can be locked in using constraints after the equilibrium has been reached, and a terminal value function can be added to account for control costs and damages accrued after the final time period. We add the following constraints to the model defined above to lock in the equilibrium solution:

$$y_{i,j,t} = y_{i,j,t\_mid} \qquad \forall (i,j) \in C, t \in T, t > t\_mid\,; \tag{8}$$

$$z_{i,j,k,l,t} = z_{i,j,k,l,t\_mid} \qquad \forall (i,j,k,l) \in N, t \in T, t > t\_mid\,; \text{ and} \tag{9}$$

$$x_{i,j,t} = x_{i,j,t\_mid} \qquad \forall (i,j) \in C, t \in T, t > t\_mid\,; \tag{10}$$

where 1<*t_mid*<*Tmax*. We choose *t_mid* and *Tmax* large enough for equilibrium to have been reached and maintained by time *t*<*t_mid*. We calculate the terminal value as the net present value of steady-state control costs and damages from time *T+1* to infinity:

$$\sum_{t=T+1}^{\infty} \beta_t * \left(\sum_{(i,j)\in C} x_{i,j,T} d + \sum_{(i,j)\in C} y_{i,j,T} e + \sum_{(i,j,k,l)\in N} z_{i,j,k,l,T} b \right) \tag{11}$$

and include this value in the objective function (1).

4.4 Model Implementation

We program this problem in Zimpl (Zuse Institute Mathematical Programming Language, version 2.08) and solve it using SCIP (Solving Constraint Integer Programs, version 1.1.0).[7] To reduce the number of parameters, we scale damages d to 1 and measure costs b and e as units of damage; this rescaling imposes no loss of generality. We use *Tmax*=100 and *t_mid*=50 for all optimizations because these time horizons are long enough for the infinite horizon steady-state to be reached and maintained for all the invasion scenarios we consider.

We use numerous comparative spatial-dynamic "experiments" to elucidate the role of economic parameters and invasion and landscape characteristics in determining the optimal control strategy. We focus on our novel results examining spatial features of the landscape and invasion, including landscape shape as well as initial invasion location and shape. For each focal characteristic, we ran optimizations for different levels of the characteristic while holding all other aspects of the invasion constant. We solved optimizations for a wide array of starting conditions, and we present a subset of these optimizations to illustrate key findings, including spatial strategies for controlling invasions.

5. Results

Optimal control strategies for invasions vary dramatically across invasion, landscape, and economic characteristics. Optimal policies range from no control to complete eradication. In between these two extremes, optimal policies include eradication of part of the invasion and containment or abandonment of the rest, immediate and complete containment, partial containment that allowed some spread prior to complete containment, and partial containment followed by abandonment of control efforts. For all examined scenarios, if clearing or eradication efforts are employed, they are optimally completed in the first time step.

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[7] SCIP is a framework for constraint integer programming based on the branch-and-bound procedure to solve optimization problems (Achterberg 2008). Branching divides the initial problem into smaller subproblems that are easier to solve, and the best solution found in the subproblems yields the global optimum. Bounding avoids enumeration of all (exponentially many) solutions of the initial problem by eliminating subproblems whose lower (dual) bounds are greater than the global upper (primal) bound.

5.1 Some Expected Results

Although not the focus of this paper, we examine how economic parameters, size of the potential invasion range, and control delays affect optimal control policies. Our findings support those in existing literature, as reviewed by Epanchin-Niell and Hastings (2010). For example, high control costs, low damages, and high discount rates reduce the amount of optimal control.[8] All else equal, invasions that have a larger potential for spread demand greater control because they face higher potential damages. Larger invasions are less likely to be optimally controlled, implying that inadvertent delay of control (e.g., by late discovery) reduces the likelihood that eradication or containment will be optimal. In addition, the net present value of costs and damages increases with control delays, highlighting the importance of finding and controlling invasions early.

5.2 Landscape Shape

Landscape shape, an important feature of landscape geometry, has significant effects on the optimal policy of an invasion because invasion range boundaries affect the costs of invasion control and damages by constraining invasion spread. Figure 1 illustrates the optimal control policies for a 2x2 cell invasion spreading in three different-shaped landscapes. All three rectangular landscapes have equal area (256 cells) but vary in length and width. The figure shows that eradication or containment is optimal across a larger range of economic parameters for invasions occuring in the compact (square) landscape (Figure 1a) than in increasingly narrow landscapes (Figure 1b, 1c). The boundaries of narrow landscapes confine the spread of species more than compact landscapes, so damages accrue more slowly, resulting in lower potential total damages.

The landscape's particular shape beyond length and width also affects optimal control policies. For example, constrictions and expansions in the landscape can alter optimal control strategies by affecting the cost of controlling the invasion (Figure 2) or the rate the invasion spreads (Figure 3).

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[8] Because we normalized marginal damages to one and scaled control costs accordingly, the effect of an increase in marginal damages is represented in our optimizations as a reduction in control costs. Specifically, a doubling of per-unit damages is modeled as halving border control and removal costs.

Figure 2 illustrates how landscape geometry can be employed strategically to optimally reduce long-term containment costs. In this scenario, complete containment in the first time period is not optimal because the extent of the exposed invasion edge (11 cell edges) is large. Instead, optimal policy slows the growth of the invasion along the center of the invasion front, delaying damages centrally, and then contains the invasion in perpetuity when it reaches the landscape constriction. This control policy slows the invasion along the region of the invasion front that has the greatest potential long-term growth of damages (because it is spreading toward the largest extent of uninvaded area) and delays complete containment until landscape features constrain long-term costs.

Landscape geometries that include areas with potentially large rates of damage accumulation, as illustrated in Figure 3, also can lead to interesting strategic containment of an invasion. In this scenario, the invasion is spreading along a narrow section of the landscape toward a region where the landscape becomes wider (and future damages from spread become larger). The narrow section of the landscape confines the invasion to spread at a rate of four cells per time period, and neither containment nor eradication is optimal because the costs of control are high relative to the avoided damages. However, if the invasion were to spread beyond the narrow region of the landscape, the rate of damage accumulation would increase rapidly because the invasion would spread in three directions rather than one. Consequently, optimal policy contains the invasion when it reaches the end of the constricted region, at which point the containment costs remain the same but the avoided damages increase.

5.3 Invasion Location

Figure 4 illustrates optimal control policies for invasion of a single cell in a 15x15 landscape at 3 different locations. From top to bottom, the panels in Figure 4 represent an invasion beginning at the center, at an edge, and in the corner of the landscape. In this example, containment is optimal across a larger range of border control costs for invasions occurring more distally, while eradication is optimal across a greater range of marginal eradication costs for invasions occurring more centrally. However, the relationship between invasion location and control costs varies across scenarios we examine.

The effect of invasion location on optimal control policy is ambiguous because invasion location affects long-term damages and costs of control in opposing ways. An invasion beginning near an edge takes longer to fully invade the landscape than an invasion that begins near the center because the furthest reaches of the landscape are more distant and the growth of the invasion is constrained by the landscape boundaries. Thus, although an uncontrolled invasion will eventually spread throughout the landscape regardless of its starting location, the net present value of potential damages from an invasion beginning near an edge are lower, which reduces the range of total control costs for which eradication or containment is optimal.

On the other hand, invasions that occur along an edge of the landscape have lower containment costs because the landscape boundaries prevent spread along the bounded edge at no cost, mediating the effects of lower damages on optimal policy. Across all optimizations, for invasions with similar characteristics, the net present value of costs and damages is higher for central invasions than for invasions that begin distally because central invasions have higher potential damages and higher control costs than do distal invasions.[9]

Invasion location can influence control costs even if the invasion does not begin immediately adjacent to a landscape boundary. Figure 5 shows the spread of an optimally controlled invasion across four time steps, demonstrating how landscape boundaries are strategically employed. The initial invasion (*t*=0) is a 4x4 block of cells located 2 cell widths from the corner of a 15x15 cell landscape. Optimal control policy contains the invasion along its central edges, while allowing the invasion to spread toward the corner of the landscape. In time periods 1–4, control is applied along the 12, 10, 8, and 10 most central borders of the invaded region, respectively, after which the invasion is contained in perpetuity. This strategy, which confines the invasion using landscape boundaries, reduces the number of exposed borders from 16 to 12, reducing periodic containment costs by 25 percent for the long term. In contrast, for an identical invasion located centrally in the landscape, immediate containment is optimal because landscape boundaries cannot be employed to reduce long-term containment costs, and total costs and damages are higher (3,696 versus 3,176). This provides another illustration of how invasion location affects optimal control policies and the net present value of costs and damages.

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[9] In general, the relative effect of invasion location on damages and control costs is determined largely by the size of the potential landscape and the extent of the invasion that is confined by landscape boundaries. The difference in potential damages from central versus distal invasions is greater in larger landscapes, and more confined invasion edges increase the range of border control costs for which containment is optimal. The range of costs for which eradication is optimal is much less mediated by the adjacency of landscape boundaries because eradication costs depend primarily on the amount of area invaded rather than the amount of edge. Thus, in most cases central invasions are optimal to eradicate across a larger range of eradication costs than invasions occurring distally (Figure 4). Furthermore, the cost-effectiveness of eradication relative to containment tends to be higher for more central invasions, because the costs of containing central invasions cannot be reduced by landscape boundaries. This result is illustrated by the steeper line dividing eradication and containment in Figure 4 for the central invasion.

Figure 6 provides an example of the effect of invasion location on optimal policy for a two-patch invasion: a corner patch in the uppermost left-hand cell of the figure and another located one cell width from the opposite corner. A large number of optimal control policies are possible for this invasion depending on the economic parameters. For example, eradication of both patches, containment of both patches, or abandonment of control are optimal policies for situations with low eradication costs, low border control costs, and high control costs, respectively. However, because of the different locations of the two patches relative to the borders, optimal policy applies dramatically different types of control to each patch for small variations in cost parameters.

For example, the lower right-hand patch is more costly to contain because it has four exposed edges; in some circumstances it is optimal to eradicate that patch and perpetually contain the patch that has only two exposed edges (Figure 6a). For the same invasion with slightly higher eradication costs, the optimal policy switches so that initial containment of the upper left-hand patch is still optimal, but the patch with more exposed borders is neither contained nor cleared (Figure 6b). Because the invasion is not fully controlled, the invasion spreads, reducing the benefits of containing the upper left-hand cell, and eventually all control efforts are optimally abandoned.

Although the landscape boundaries cannot be used to reduce the amount of exposed edge on the lower right-hand patch in Figure 6b, the optimal policy applies control to the lower right-hand patch to slow the advance centrally and direct growth toward the corner for the first two periods. This approach, which was also employed for the invasion in Figure 2, reduces the long-term damages from the invasion by delaying spread in the direction with the highest potential growth. Similarly, in Figure 5, control efforts slow the spread of the invasion into the corner and prevent spread centrally (where the long-term potential growth is greatest). These examples illustrate how invasion location within the landscape can affect optimal spatial allocation of control.

5.4 Invasion Shape and Contiguity

Even holding size and location constant, the shape and contiguity of an invasion affect optimal levels and spatial allocation of control effort. For example, containing a compact invasion, which has a lower edge-to-area ratio, is optimal over a wider range of border control costs than containing a similar patchy invasion. This is illustrated in Figure 7, which shows the optimal control strategies for a compact 2x2 cell invasion and a patchy invasion of 4 equidistant cells near the center of a 15x15 cell landscape. Optimal policy mandates abandoning control of the patchy invasion across a larger range of marginal control costs because it has higher containment costs.[10] Because containment is more costly for patchy invasions, eradication is optimal across a larger range of marginal eradication costs for patchy invasions than for compact invasions. This is evidenced in Figure 7 by the steeper slope of the line dividing containment and eradication for the patchy invasion.

Clearly, an important feature of invasion geometry is its influence on the invasion edge and the effect of edge length on containment costs. Figures 2 and 5 show that employing landscape boundaries can reduce the extent of exposed edge. In other scenarios, optimal policies clear cells or allow spread before containment to alter the shape of an invasion and reduce the length of edge. For example, Figure 8 shows an initially "edgy" invasion for which optimal policy combines removal and spread prevention to reduce the number of exposed edges from 11 to 8 before complete containment.

Finally, with respect to patchy invasions, our scenarios show that optimal control strategies can vary across patches of invasion within a landscape and control strategies for individual patches depend on the entire landscape context (e.g., Figures 6 and 9). Just as dynamic problems involve choosing an entire time path of interdependent decisions, optimal control of a spatial-dynamic system involves simultaneously choosing control efforts across spatially separated patches because the avoided future damages of controlling each patch depend on the control efforts and spread rates at other patches. For example, Figure 9 shows an invasion for which optimal policy requires eradication of one patch and slowing, followed by abandonment, of the other. However, for an identical invasion with slightly higher border-control costs (b=16), the benefits of slowing the spread of the large patch are reduced so that the gains from eradication also are reduced, and eradication of the small patch ceases to be optimal.

6. Synthesis and Discussion

Our analysis shows that many spatial characteristics of an invasion determine the optimal policy, including such novel factors as landscape shape, invasion shape, and location. Nearly identical invasions can have dramatically different optimal control policies if they differ in any

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[10] This occurs even though long-term damages from abandoning control are slightly higher for the patchy invasion than for the compact invasion because the patchy invasion spreads faster. It is also important to note that across the range of border control costs for which containment of the compact invasion is mandated, optimal management for the patchy invasion may involve containment, spread followed by containment, slowing, and even abandonment.

of these characteristics. An unfortunate consequence is that deriving clear and simple rules for how to best manage all invasions is unlikely. However, we show how these factors affect the qualitative nature of optimal control policies and provide intuition for these results.

6.1 Landscape Shape

Our results show that invasions in more compact landscapes generally warrant more control because spread is less constrained, resulting in higher long-term damage potential. However, landscape shape also affects the likelihood that an invasion will be located near enough to boundaries to reduce long-term containment costs. Nonconvexities in the landscape, such as constrictions and expansions, influence optimal control policies by affecting the costs of containment and invasion spread rates in those regions. Interestingly, the presence of landscape nonconvexities are the only situation we found for which delaying the start of control efforts can be optimal.[11]

6.2 Invasion Location

The initial location of an invasion affects optimal control and total costs and damages of an invasion. Central invasions face higher potential damages because the invasion can spread through the landscape more rapidly, and control costs may be lower for invasions that begin distally if landscape boundaries can help contain the invasion. Thus, centrally located invasions tend impose higher total costs and damages than do distal invasions. Location also influences the optimal spatial allocation of control by determining the direction of greatest potential invasion spread.

6.3 Invasion Shape and Contiguity

The shape of an invasion also influences optimal control policies because it affects containment costs and spread rates in three important ways. First, a greater amount of invasion edge decreases the range of control costs for which containment is optimal, shifting policies

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[11] Several other studies have found that delaying control can be optimal, but for different reasons than shape. Burnett et al. (2007) and Olson and Roy (2008) find that when control costs are stock dependent (i.e., higher marginal control costs for smaller invasions), delaying control to reduce control costs can be optimal. For example, Burnett et al. suggest that delaying control is optimal for *Miconia* invasions on some of the Hawaiian islands. We are unaware of any other studies than ours that have explicitly examined the effects of landscape shape on optimal control policy.

toward eradication or abandonment. Second, for edgy invasions, spread or removal before containment to reduce the amount of exposed edge can reduce long-term control costs. Third, optimal control of patchy invasions depends on the entire landscape, and control efforts can vary across patches based on patch and total invasion characteristics.

6.4 Spatial Aspects of Control

Land managers can strategically use landscape boundaries to reduce long-term containment costs, further highlighting the role of landscape geometry in controlling invasions. These boundaries, which are determined by the habitability and porousness of the landscape to the invader, include elevational, temperature, and precipitation gradients, soil types, and water bodies. Land managers should account for them when determining optimal control.[12]

Results regarding the effect of invasion shape on optimal control effort provide insight on improving the use of barrier zones, which have been used to control a variety of invasions, including the boll weevil and gypsy moth. This approach applies control efforts along the growing edge of an invasion to slow its spread (Sharov and Liebhold 1998). Our findings suggest that applying control efforts homogenously along the growing edge can be suboptimal. Instead, applying control to reduce the length of the invasion edge can be better in some cases. Also, greater amounts of control should be applied to slow the invasion in the direction of greatest long-term potential growth.

High-density, established invasions can grow quickly and create new satellite populations. An unanswered question is whether control efforts should be focused on large, core patches or smaller, satellite patches.[13] In this study, we do not consider long-distance dispersal processes or differential densities among invaded patches, but our results support two points. First, greater amounts of control tend to be optimal for smaller invasions because eradication and

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[12] For example, an elevationally constrained invasion spreading in a valley between two mountain ranges may best be contained in a narrow region of the valley. Also, it may be possible to employ restoration to create landscape barriers, because many species more easily invade disturbed ecosystems than diverse or undisturbed systems (e.g., Maron and Marler 2007). Creating barriers through restoration or habitat elimination in strategic areas of the landscape may reduce the long-term costs of containing invasions that are too widespread to eradicate.

[13] This questions was first addressed in the literature by Moody and Mack (1988). Taylor and Hastings (2004) point out that even theoretical frameworks suggest different prioritizations: “the population biology approach suggests that, in general, outliers contribute the most to range expansion and should be removed first, whereas the metapopulation approach suggests prioritizing core populations that supply most of the new propagules.” Epanchin-Niell and Hastings (2010) also review literature addressing this question.

containment costs are lower, suggesting that it may be optimal to focus more control effort on smaller, satellite invasions in some cases. Second, optimal control for each patch of an invasion depends on the entire invasion and landscape, so patches cannot be considered independently. A blanket strategy or prioritization is thus unlikely to be optimal.

The management of many invasive plants is not regulated because they are classified as too widespread to justify eradication. Our results show that under some circumstances, however, it is optimal to eradicate one patch of an invasion even while allowing other patches to spread. Thus it may be worth controlling small populations that occur far from the main invasion, even when an invasion is widespread. Furthermore, it may be optimal to slow or contain widespread invasions, even when eradication is not justified, especially when large potential for further spread exists.

6.5 Some Principles of Optimal Bioinvasion Control

The following basic principles arise from this study:

- Protect large areas of uninvaded landscape. Invasions with greater potential for spread, for example, portend larger future damages and thus warrant greater control effort.
- Reduce the extent of exposed invasion front to reduce long-term containment costs. Methods include employing landscape features or altering the shape of the invasion through spread or removal.
- Slow the spread of an invasion in the direction of greatest potential local or long-term growth.
- Do not delay eradication.

6.6 Assumptions and Generalizability

As with any model, the one we employ makes simplifying assumptions so that the problem of bioinvasion control is computationally tractable. Our model captures important features of growth and spread, within which we represent relevant spatial features of invasions and solve for fully explicit spatial-dynamic optimal policies. One key feature of invasions not incorporated into our model is uncertainty. In particular, our model does not include uncertainty due to lack of information (e.g., about spread rates, damages, costs) or stochastic processes (e.g., in spread, damages, costs). Nonetheless, our deterministic framework provides important intuition about these more general, uncertain invasion cases.

Some recent work on invasions has relaxed the assumption of perfect information to examine issues of adaptive, robust invasion management and the value of information, primarily in the context of nonspatial control of pest density (e.g., Carrasco et al. 2009; Eiswerth and van Kooten 2007; Saphores and Shogren 2005; Yokomizo et al. 2009). Our work contributes insight to this branch of research by suggesting that space may play an important role in adaptive management and learning. In particular, our model shows that the costs and damages of allowing invasion to spread can vary depending on direction, even in a homogeneous region, simply due to differences in potential long-term growth and containment. Thus, during the learning process about a novel invader, it might be optimal to allow the invasion to spread specifically in the direction of natural barriers or low potential long-term growth, where the damages and control costs are likely the lowest.

Stochasticity in spread processes, control effectiveness, costs, and damages are also potentially relevant to invasion management. The main limitation of our deterministic model is its inability to allow for stochastic, rare, long-distance dispersal events. Unfortunately, this is a very difficult problem to address in the context of explicit space and is yet unsolved. However, our model does provide some intuition. For species that exhibit long-distance dispersal, we expect eradication to be optimal across a greater range of economic parameters because damages would accrue faster than they would if dispersal was over a shorter distance. Also, we believe that containment would be optimal across a smaller range of economic parameters, because containment costs would be higher as satellite invasion patches were removed or the benefits would be lower as the invasion established beyond the containment zone. In contrast, we expect a shift away from eradication and containment in regions that incur repeat invasions because the benefits of both types of control are reduced.

Nonetheless, we believe that the same qualitative patterns we found in this study with respect to economic, landscape, and invasion characteristics, as well as spatial strategies for control, will hold for invasions with different patterns of spread. As in our deterministic model, we expect that optimal control will favor the maintenance or formation of compact and landscape-constrained invasions to minimize the local containment costs and the potential for long-distance dispersal. In addition, we expect greater control efforts to be applied in the direction of high potential long-term damages. For an invasion of a species exhibiting long-distance dispersal and spreading in a narrow, constrained region that opens into a more expansive area, we expect control efforts that employ the landscape constriction to facilitate control but begin earlier. With respect to stochasticity in costs and damages, our deterministic model transfers easily because parameters can be viewed as expected costs and damages for a risk-neutral social maximizer.

Another necessary modeling assumption regards the shape of the cost and damage functions. We employ constant marginal costs and damages, but alternative assumptions can apply depending on the particular invasive species (Epanchin-Niell and Hastings 2010). We chose what we believe is a general but tractable assumption and have tried to capture features of a wide diversity of potential invaders by optimizing over various cost and damage parameters. In addition, our specification of two separate cost parameters—one for clearing and one for preventing spread—allows particular flexibility for capturing biological and economic characteristics of particular species.

Finally, we did not present results showing how heterogeneous costs, damages, spread rates, and other factors affects optimal control. However, our results provide some guidance as to how such heterogeneities might affect control. For example, if damages from the invasion are higher in certain regions of the landscape, control efforts will be directed to prevent or delay the spread of the invasion into those regions. Areas with lower rates of spread might be employed to contain an invasion at lower costs.

7. Conclusions

In this paper, we aim to provide useful and novel understanding of economically optimal control of bioinvasions. Employing a two-dimensional, spatially explicit biological spread model allows us to examine control strategies that vary across space and time. It also allows us to identify how the geometry of the invasion and landscape affect the qualitative nature of optimal control policies. As we show, the optimal solution for a spatially explicit optimization problem generates a far richer set of solution characteristics and more nuanced conclusions about how to control bioinvasions than work that treats space only implicitly or does not allow for differentiated control across space. We describe and provide intuition for the wide spectrum of optimal solutions that emerge as we perturb bioeconomic parameters as well as landscape and invasion geometry.

The second purpose of this paper is to use bioinvasions as a model case study for learning about the wider class of spatial-dynamic problems. Economics has a rich legacy of analysis that addresses dynamic problems and the spatial nature of economic activities. In contrast, problems driven by spatial-dynamic processes have only recently begun to receive attention.[14] Spatial-dynamic problems are characterized by diffusion or spread processes that generate patterns over space and time. Examples aside from bioinvasions include groundwater contamination, epidemics, forest fires, migration and movement, and technology adoption. In human-mediated landscapes, economic agents may be affected by these spatial-dynamic processes, and they also may affect the patterns that unfold. In a setting in which agents are located over space, a general question arises about how to control spatial-dynamic processes in a manner that maximizes welfare across the whole landscape. As we have shown for the bioinvasion case, where spread generates damages, the issue is not only when and at what level of intensity to initiate controls, but also where.[15] We have found, in general, that the dynamic parts of the solution—those concerned with when and at what level of intensity to initiate controls—are intertwined in complex ways with the spatial part of the solution, or where to initiate controls.

When we account for space and time, some results reflect our intuition about the dynamic components of the problem, while other features are novel. Most importantly, adding space necessitates concern about geometric characteristics of problems in addition to concern about more familiar metrics such as size or quantity. To highlight some of our new findings about bioinvasions that may shed light on the larger class of spatial-dynamic problems, we compare general principles that apply to dynamic problems with some new results that emerge from our consideration of spatial dynamics:

- In dynamic problems, time is the index that differentiates decisions and runs only forward. Time and space identify decisions in spatial-dynamic problems, which run forward in time but can spread and contract in multiple directions over space.
- The optimal solutions to interesting dynamic problems always involve a trade-off between contemporaneous benefits (costs) and the present value of long-term costs (benefits). Importantly, the solution is forward-looking, scanning the complete time

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[14] This includes work on marine systems by Clark (1973), Brown and Roughgarden (1997), Sanchirico and Wilen (1999, 2005, 2007), Janmaat (2005), Costello and Polasky (2008), Brock and Xepapadeas (2008), Ding and Lenhart (2009), and Smith et al. (2009); a study on spatial-endogenous fire risk and fuel management by Konoshima et al. (2008); work on bioinvasions by Wilen (2007) and studies described previously in this article; work on managing dispersive pests by Bhat et al. (1993, 1996) and Huffaker et al. (1992); work on disease management, including by Rich and Winter-Nelson (2007), Rich et al. (2005a, 2005b), and Ding et al. (2007); and an application to managing a semi-arid system by Brock and Xepapadeas (2010).

[15] When spatial-dynamic processes generate benefits, similar issues arise in choosing policies that encourage, rather than inhibit, spread.

horizon, adding up the marginal impacts over that horizon, and comparing those anticipated impacts with current marginal costs. Spatial-dynamic problems also are forward-looking but over time *and* space. Optimal bioinvasion controls in our problem account for the size and character of the potential space (and hence damages) that lies ahead in time and space of the advancing invasion front. Directionally differentiated damages influence the degree of control exerted at any point in time and space. Large prospective damages (either from a large amount of space or from high damages per unit of space) in the path of a spreading front will call forth higher levels of control early and at locations often roughly orthogonal to the path of the front.

- Dynamic optimization solutions critically depend on the initial state of the system, generally measured by the size of capital or resource level at some starting date. For example, the smaller the initial capital level relative to its steady-state level, the larger current optimal investment should be. For spatial-dynamic problems, the geometry of the initial state matters as much as its size. As we show, small variations in shape and location in the landscape can lead to qualitatively different optimal solutions. For example, whether eradication or containment may be optimal depends on not only basic costs, damages, size of invaded area, and discount rate, but also how large the initial invasion is relative to the landscape, where it is located, the extent of exposed invasion edge, and other spatial features.

Economists will need to develop new intuition about spatial-dynamic problems by analyzing these and other cases before we can understand what features of the solutions to this class of problems appear to be general. In the meantime, these are just a few of the characteristics that we conjecture may emerge as general properties of solutions of other spatial-dynamic optimization problems.

References

Achterberg, T., T. Berthold, T. Koch, and K. Wolter. 2008. Constraint integer programming: A new approach to integrate CP and MIP. In L. Perron, and M. Trick (eds.), Integration of AI and OR Techniques in Constraint Programming for Combinatorial Optimization Problems. Berlin: Springer, 6–20.

Albers, H., C. Fischer, and J. Sanchirico. 2010. Invasive species management in a spatially heterogeneous world: Effects of uniform policies. Resource Energy Econ. 32(4): 483–499.

Bhat, M.G., R.G. Huffaker, and S.M. Lenhart. 1993. Controlling forest damage by dispersive beaver populations: Centralized optimal management strategy. Ecol. Appl. 3(3): 518–530.

Bhat, M.G., R.G. Huffaker, and S.M. Lenhart. 1996. Controlling transboundary wildlife damage: Modeling under alternative management scenarios. Ecol. Model. 92(2–3): 215-224.

Blackwood, J., A. Hastings, and C. Costello. 2010. Cost-effective management of invasive species using linear-quadratic control. Ecol. Econ. 69(3): 519–527.

Brock, W., and A. Xepapadeas. 2004. Spatial analysis: Development of descriptive and normative methods with applications to economic-ecological modelling. SSRI Working Paper #2004-17. Madison, WI: University of Wisconsin Department of Economics.

Brock, W., and A. Xepapadeas. 2008. Diffusion-induced instability and pattern formation in infinite horizon recursive optimal control. J. Econ. Dynam. Control 32(9): 2745–2787.

Brock, W., and A. Xepapadeas. 2010. Pattern formation, spatial externalities and regulation in coupled economic-ecological systems. J. Environ. Econ. Manage. 59(2): 149–164.

Brown, G., and J. Roughgarden. 1997. A metapopulation model with private property and a common pool. Ecol. Econ. 22(1): 65-71.

Burnett, K., B. Kaiser, and J. Roumasset. 2007. Economic lessons from control efforts for an invasive species: Miconia calvescens in Hawaii. J. For. Econ. 13(2–3): 151–167.

Carrasco, L., R. Baker, A. Macleod, J. Knight, and J. Mumford. 2009. Optimal and robust control of invasive alien species spreading in homogeneous landscapes. J. R. Soc., Interface 7(44): 529–540.

Clark, C. 1973. Profit maximization and the extinction of animal species. J. Polit. Economy 81(4): 950–961.

Costello, C., and S. Polasky. 2008. Optimal harvesting of stochastic spatial resources. J. Environ. Econ. Manage. 56(1): 1–18.

Deen, W., A. Weersink, C. Turvey, and S. Weaver. 1993. Weed control decision rules under uncertainty. Rev. Agr. Econ. 15(1): 39–50.

Ding, W., and S. Lenhart. 2009. Optimal harvesting of a spatially explicit fishery model. Nat. Resour. Model. 22(2): 173–211.

Ding, W.D., L.J. Gross, K. Langston, S. Lenhart, and L.A. Real. 2007. Rabies in raccoons: Optimal control for a discrete time model on a spatial grid. J. Biol. Dyn. 1(4): 379–393.

Eiswerth, M.E., and W.S. Johnson. 2002. Managing nonindigenous invasive species: Insights from dynamic analysis. Environ. Resour. Econ. 23(3): 319–342.

Eiswerth, M.E., and G.C. van Kooten. 2007. Dynamic programming and learning models for management of a nonnative species. Can. J. Agric. Econ. 55(4): 485–498.

Elith, J., C. Graham, R. Anderson, M. Dudik, S. Ferrier, A. Guisan, R. Hijmans, F. Huettmann, J.R. Leathwick, A. Lehmann, J. Li, L.G. Lohmann, B.A. Loiselle, G. Manion, C. Moritz, M. Nakamura, Y. Nakazawa, J.M. Overton, A.T. Peterson, S.J. Phillips, K. Richardson, R. Scachetti-Pereira, R.E. Schapire, J. Soberon, S. Williams, M.S. Wisz, and N.E. Zimmermann. 2006. Novel methods improve prediction of species' distributions from occurrence data. Ecography 29(2): 129–151.

Epanchin-Niell, R.S., and A. Hastings. 2010. Controlling established invaders: integrating economics and spread dynamics to determine optimal management. Ecol. Lett. 13(4): 528–541.

Farias, V., D. Saure, and G. Weintraub. Forthcoming. An Approximate Dynamic Programming Approach to Solving Dynamic Oligopoly Models. In review.

Grevstad, F.S. 2005. Simulating control strategies for a spatially structured weed invasion: Spartina alterniflora (Loisel) in Pacific Coast estuaries. Biol. Invasions 7(4): 665–677.

Hanski, I. 1999. Metapopulation Ecology. Oxford, UK: Oxford University Press.

Higgins, S., D. Richardson, and R. Cowling. 2000. Using a dynamic landscape model for planning the management of alien plant invasions. Ecol. Appl. 10(6): 1833–1848.

Hof, J. 1998. Optimizing spatial and dynamic population-based control strategies for invading forest pests. Nat. Resour. Model. 11: 197–216.

Huffaker, R., M. Bhat, and S. Lenhart. 1992. Optimal trapping strategies for diffusing nuisance-beaver populations. Nat. Resour. Model. 6: 71–97.

Janmaat, J.A. 2005. Sharing clams: tragedy of an incomplete commons. J. Environ. Econ. Manage. 49(1): 26–51.

Kawasaki, K., F. Takasu, H. Caswell, and N. Shigesada. 2006. How does stochasticity in colonization accelerate the speed of invasion in a cellular automaton model? Ecol. Res. 21(3): 334–345.

Konoshima, M., C.A. Montgomery, H.J. Albers, and J.L. Arthur. 2008. Spatial-endogenous fire risk and efficient fuel management and timber harvest. Land Econ. 84(3): 449–468.

Maron, J., and M. Marler. 2007. Native plant diversity resists invasion at both low and high resource levels. Ecology 88: 2651–2661.

Moody, M.E., and R.N. Mack. 1988. Controlling the spread of plant invasions: The importance of nascent foci. J. Appl. Ecol. 25(3): 1009–1021.

Olson, L. 2006. The economics of terrestrial invasive species: a review of the literature. Agr. Resource Econ. Rev. 35(1): 178–194.

Olson, L. and S. Roy. 2002. The economics of controlling a stochastic biological invasion. Am. J. Agric. Econ. 84(5): 1311–1316.

Olson, L., and S. Roy. 2008. Controlling a biological invasion: A non-classical dynamic economic model. Econ. Theory 36(3): 453–469.

Pannell, D.J. 1990. An economic response model of herbicide application for weed-control. Aust. J. Agric. Econ. 34(3): 223–241.

Peterson, A. 2003. Predicting the geography of species' invasions via ecological niche modeling. Q. Rev. Biol. 78: 419–433.

Potapov, A.B., and M.A. Lewis. 2008. Allee effect and control of lake system invasion. Bull. Math. Biol. 70(5): 1371–1397.

Rich, K.M., and A. Winter-Nelson. 2007. An integrated epidemiological-economic analysis of foot and mouth disease: Applications to the Southern Cone of South America. Am. J. Agric. Econ. 89(3): 682–697.

Rich, K.M., A. Winter-Nelson, and N. Brozovic. 2005a. Regionalization and foot-and-mouth disease control in South America: Lessons from spatial models of coordination and interactions. Quart. Rev. Econ. Finance 45(2–3): 526–540.

Rich, K.M., A. Winter-Nelson, and N. Brozovic. 2005b. Modeling regional externalities with heterogeneous incentives and fixed boundaries: Applications to foot and mouth disease control in South America. Rev. Agr. Econ. 27(3): 456–464.

Sanchirico, J., and J. Wilen. 1999. Bioeconomics of spatial exploitation in a patchy environment. J. Environ. Econ. Manage. 37(2): 129-150.

Sanchirico, J., and J. Wilen. 2005. Optimal spatial management of renewable resources: Matching policy scope to ecosystem scale. J. Environ. Econ. Manage. 50(1): 23–46.

Sanchirico, J., and J. Wilen. 2007. Sustainable use of renewable resources: Implications of spatial-dynamic ecological and economic processes. Int. Rev. Environ. Resource Econ. 1(4): 367–405.

Sanchirico, J.N., J.E. Wilen, and C. Coleman. 2010a. Optimal rebuilding of a metapopulation. Am. J. Agric. Econ. 92(4): 1087–1102.

Sanchirico, J., H. Albers, C. Fischer, and C. Coleman. 2010b. Spatial management of invasive species: Pathways and policy options. Environ. Resource Econ. 45(): 517–535.

Saphores, J.D.M. 2000. The economic threshold with a stochastic pest population: A real options approach. Am. J. Agric. Econ. 82(3): 541–555.

Saphores, J., and J. Shogren. 2005. Managing exotic pests under uncertainty: optimal control actions and bioeconomic investigations. Ecol. Econ. 52(3): 327–339.

Sharov, A.A., and A.M. Liebhold. 1998. Bioeconomics of managing the spread of exotic pest species with barrier zones. Ecol. Appl. 8(1998): 833–845.

Shigesada, N., and K. Kawasaki. 1997. Biological Invasions: Theory and Practice. Oxford, UK: Oxford University Press.

Smith, M., J. Sanchirico, and J. Wilen. 2009. The economics of spatial-dynamic processes: Applications to renewable resources. J. Environ. Econ. Manage. 57(1): 104–121.

Taylor, C.M., and A. Hastings. 2004. Finding optimal control strategies for invasive species: A density-structured model for Spartina alterniflora. J Appl Ecol 41(6): 1049–1057.

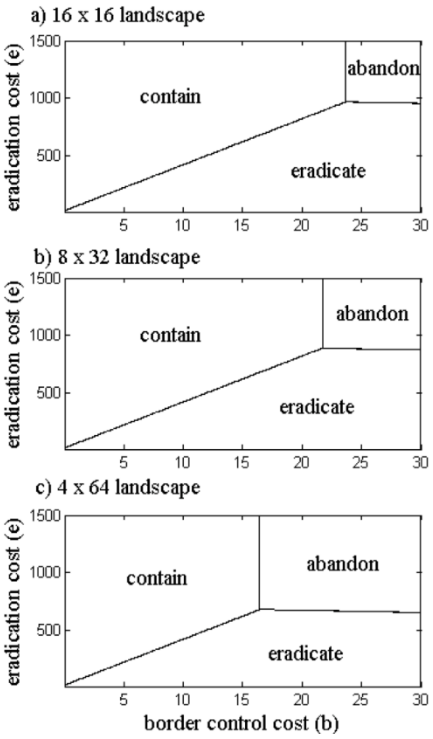
Wadsworth, R., Y. Collingham, S. Willis, B. Huntley, and P. Hulme. 2000. Simulating the spread and management of alien riparian weeds: Are they out of control? J Appl Ecol 37(Supplement s1): 28–38.

Wilen, J. 2007. Economics of spatial-dynamic processes. Am. J. Agric. Econ. 89: 1134–1144.

Yokomizo, H., H. Possingham, M. Thomas, and Y. Buckley. 2009. Managing the impact of invasive species: The value of knowing the density-impact curve. Ecol. Appl. 19: 376–386.

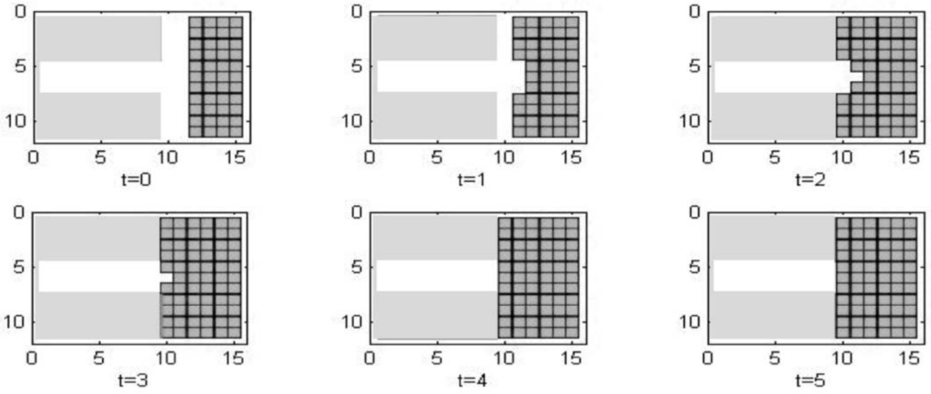
Figures

Figure 1. Optimal Control Strategies for Three Landscape Shapes



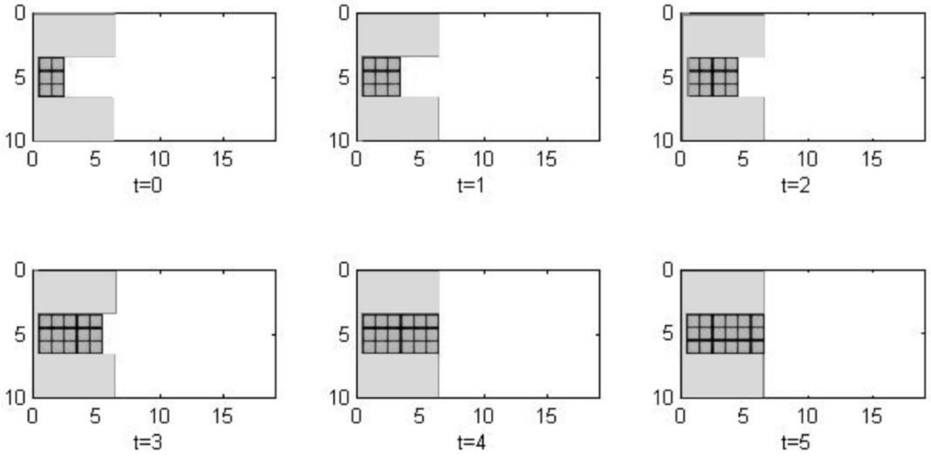
Notes: Parameter space shows how the optimal control strategy varies based on eradication and border control costs. The three equal-sized landscapes are: a) 16x16, b) 8x32, and c) 4x64. Each initially is invaded by a central 2x2 invasion. (*r* = 0.05).

Figure 2. Example of Optimal Control in a Landscape with a Constriction



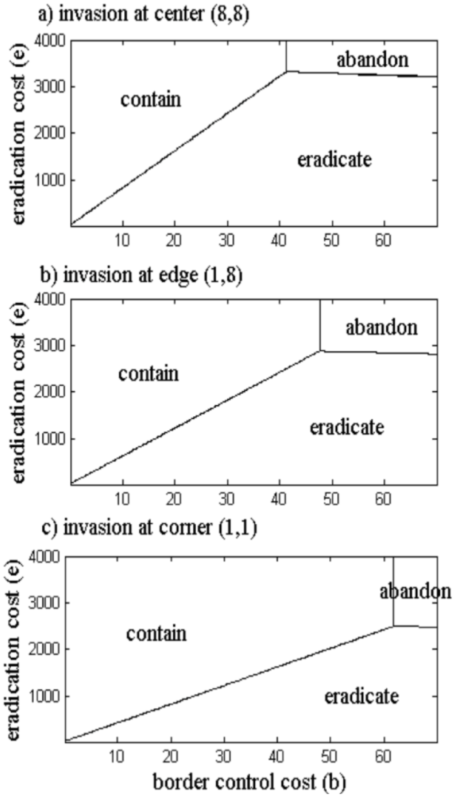
Notes: The region is 11x15 with two 4x9 sections removed (the light gray areas are not invadable). Optimal policy slows the spread of the invasion as it approaches the landscape constriction, where it is ultimately contained in perpetuity. (*r* = 0.05, *b* = 7, *e* = 250).

Figure 3. Example of Optimal Control in a Landscape with an Expansion



Notes: The region is 9x18 with two 3x6 sections removed (the light gray areas are not invadable). Optimal policy allows the invasion to spread until it reaches the end of the narrow section at time *t* = 4. Control efforts begin in time period 5 that contain the invasion in perpetuity. (*r* = 0.05, *b* = 27, *e* =250).

Figure 4. Optimal Control Strategies for Three Initial Invasion Locations



Note: Parameter space shows how the control strategy varies based on eradication and border control costs for a single cell invasion in a 15x15 cell landscape located: a) centrally, cell (8,8); b) at an edge (1,8); and c) at a corner, cell (1,1). (*r* = 0.05).

Figure 5. Optimal Control of an Invasion in a 15x15 Cell Landscape by a 4x4 Patch of Cells Near a Corner of the Landscape (*r* = 0.05, *b* = 10, *e* =230).

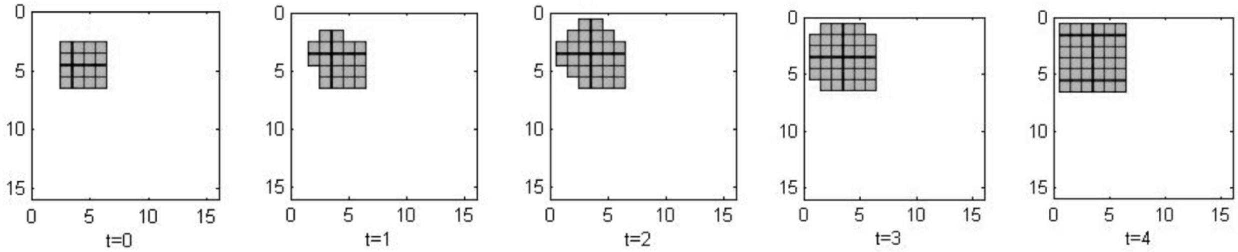
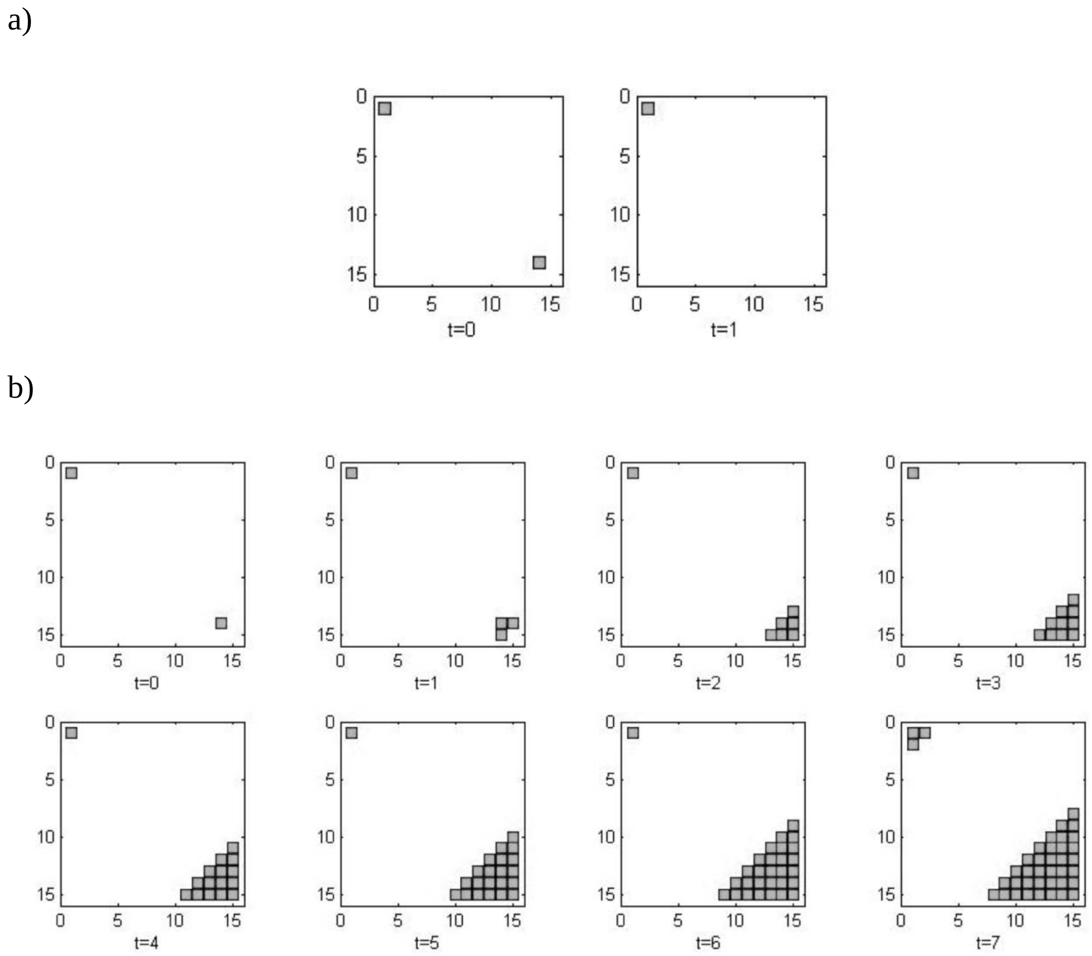
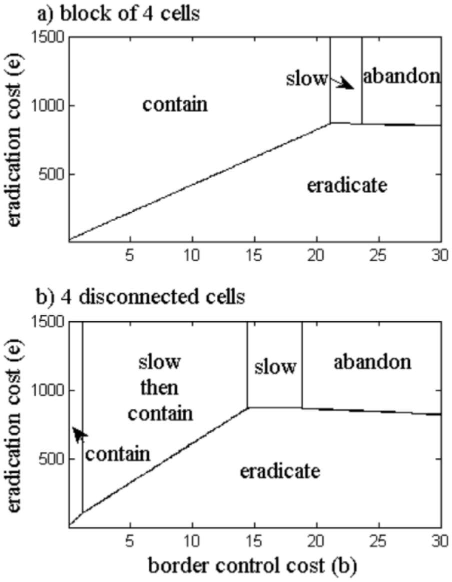


Figure 6. Optimal Control of a 2 Patch Invasion in a 15x15 Cell Landscape for 2 Different Eradication Costs



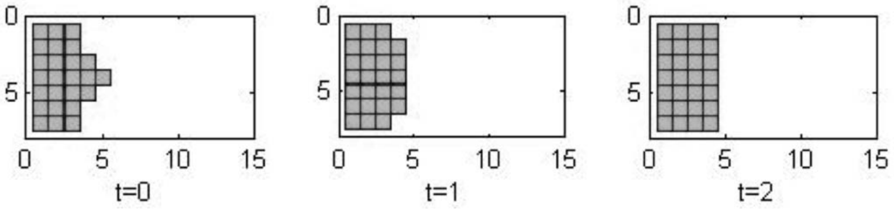
Notes: Invasion begins in cells (1,1) and (14,14) (in a corner and near a corner, respectively). In scenario (a) the optimal policy eradicates the cell in the lower right-hand corner in the first time period and contains the invasion in the upper left-hand corner in perpetuity. (r = 0.05, b = 27, e =1600). With slightly higher eradication costs in scenario (b), the optimal policy slows the invasion for the first 6 time periods and abandons control over the whole landscape at time t = 7. (r = 0.05, b = 27, e =1800).

Figure 7. Optimal Control Strategies for Two Invasions that Differ in Contiguity



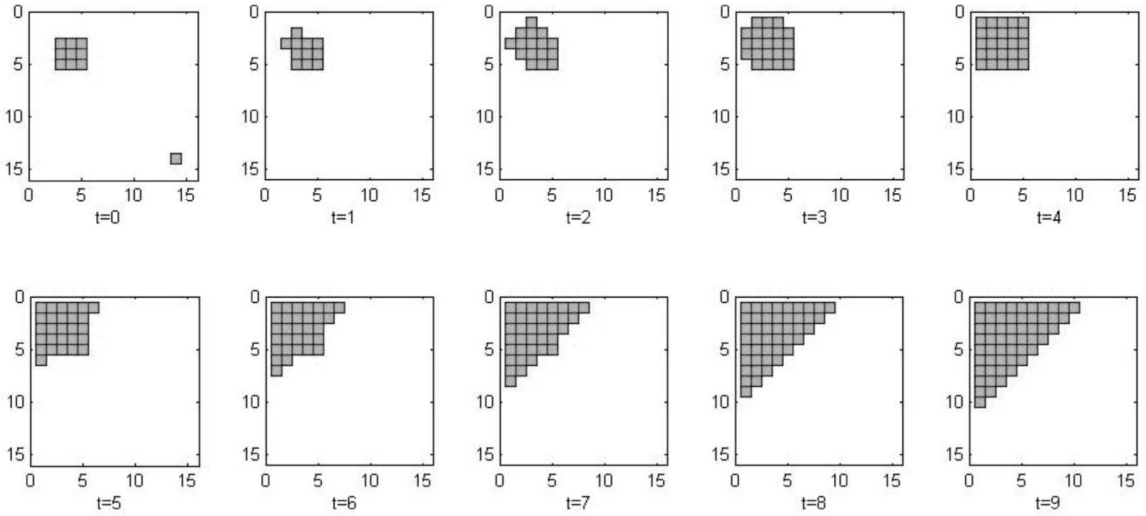
Notes: Parameter space shows how the optimal control strategy varies based on eradication and border control costs for invasion by: a) 4 contiguous or b) 4 equidistant, noncontiguous cells in a 15x15 landscape. The initial invasion occurs near the center of the landscape as: a) a block of 4 cells ((8,8), (8,9), (9,8), (9,9)), and b) 4 separated cells ((7,7), (7,9), (9,7), (9,9)). (r = 0.05).

Figure 8. Optimal Control of an Invasion in a 7x14 Cell Landscape by a Patch of Cells with Local Concavities



Note: The optimal policy eradicates one cell and slows the spread in the first time period, partially contains the invasion in t = 2, and contains the invasion in perpetuity beginning in the third time period. (r = 0.05, b = 7, e = 83).

Figure 9. Optimal Control of an Invasion in a 15x15 Cell Landscape by a Small (1 cell) and Large (9 cell) Patch



Optimal policy eradicates the small patch and slows the spread of the larger patch by directing spread into the corner of the landscape. Eventually the invasion spreads to fill the entire landscape. (*r* = 0.05, *b* = 14, *e* =450).