DISCUSSION PAPER

May 2009 ■ RFF DP 09-19

Robust Control in Global Warming Management

An Analytical Dynamic Integrated Assessment

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RESOURCES FOR THE FUTURE

Robust Control in Global Warming Management: An Analytical Dynamic Integrated Assessment*

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Abstract

Imperfect measurement of uncertainty (deeper uncertainty) in climate sensitivity is introduced in a two-sectoral integrated assessment model (IAM) with endogenous growth, based on an extension of DICE. The household expresses ambiguity aversion and can use robust control via a 'shadow ambiguity premium' on social carbon cost to identify robust climate policy feedback rules that work well over a range such as the IPCC climate sensitivity range (IPCC, 2007*a*). Ambiguity aversion, in combination with linear damage, increases carbon cost in a similar way as a low pure rate of time preference. However, ambiguity aversion in combination with non-linear damage would also make policy more responsive to changes in climate data observations. Perfect ambiguity aversion results in an infinite expected shadow carbon cost and a zero carbon consumption path. Dynamic programming identifies an analytically tractable solution to the IAM.

Keywords: climate policy, carbon cost, robust control, Knightian uncertainty, ambiguity aversion, integrated assessment

JEL classification: C73, C61, Q54

1 Introduction

An essential component subject to scientific uncertainty in climate modeling is equilibrium climate sensitivity, defined as the ratio between a steady-state change in mean atmospheric temperature ΔT_t and a steady-state change in radiative forcing ΔR_t. The IPCC Executive Summary IPCC (2007*a*) stated 'The equilibrium climate sensitivity is a measure of the climate system response to sustained radiative forcing. It is not a projection but is defined as the global average surface warming following a doubling of carbon dioxide concentrations. It is *likely* to be in the range 2.0^{o}C to 4.5^{o}C with a best estimate of about 3.0^{o}C, and is *very unlikely* to be less than 1.5^{o}C. Values substantially higher than 4.5^{o}C cannot be excluded,

*Financial support to this project from the CLIPORE program financed by MISTRA as well as the Swedish Energy Agency is gratefully acknowledged. I am grateful to Daniel Johansson at Physical Resource Theory, Chalmers University of Technology, Sweden for valuable comments on the climate modeling.

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but agreement of models with observations is not as good for those values.' Equilibrium climate sensitivity depends on several underlying physical feedback processes that affect equilibrium mean temperature which are hard to predict. Some of the most uncertain are the cloud effect, water vapor, albedo and vegetation effect, see e.g. Harvey (2000) and Hansen et al. (1984).

An analysis by Roe and Baker (2007) shows that the climate sensitivity probability distribution is highly sensitive to uncertainties in underlying physical feedback factors. Besides being skewed with a thicker high-temperature tail that is not likely to be reduced despite scientific progress in understanding (reducing variance in) underlying feedback factors, the measurement of the climate sensitivity probability distribution becomes uncertain in mean as well as variance Roe and Baker (2007). Thus we are facing *imperfect measurement of uncertainty* over climate sensitivity outcomes, while expected utility theory and risk aversion require that objective or subjective probabilities can be assigned to each outcome. Hence, the question is raised whether the traditional concepts of expected utility theory and risk aversion can explain and capture reasonings behind precautionary principles in climate change policy. In decision theory the discussion on imperfect measurement of probability distributions is not new. Already Knight (1971) suggested that for many choices the assumption of known probability distributions is too strong and therefore distinguished between 'measurable uncertainty' (risk) and 'unmeasurable uncertainty', reserving the latter denotation to include also unknown probabilities. In the literature, unmeasurable uncertainty has been named Knightian uncertainty, deeper uncertainty or simply uncertainty to distinguish it from risk. Keynes (1921), in his treatise on probability, put forward the question whether we should be indifferent between two scenarios that have equal probabilities, but the first scenario has subjective probabilities while the second has objective probabilities. Savage's Sure-Thing principle (Savage, 1954) argued that we could, while Ellsberg's experiment (Ellsberg, 1961) showed that individuals facing two lotteries, the first one with known probabilities and the second one with unknown probabilities, tended to prefer to bet on outcomes in the first lottery to bet on outcomes in the second lottery where they had to rely on subjective probabilities, thus contradicting the Sure-Thing principle. This behavior was referred to as ambiguity (or uncertainty) aversion as a broader aversion than risk aversion.

Knightian (or deeper) uncertainty and ambiguity aversion are illustrated in Table 1. Mr Gentle chooses between rows, deciding whether to buy an umbrella before (Buy Now) a walk with Mrs Gentle or to wait (Wait-n-See) and buy it only if it starts raining during the walk. Mr Gentle knows from the weather report that the probability of rain is 0.5. With 'Buy Now' follows a cost of carrying the umbrella as insurance premium, while the 'Wait-n-See' leaves the couple with unmeasurable uncertainty in probability p of finding a shop close nearby them selling umbrellas when and if it should start raining during the walk. Whether the certain 'Buy Now' choice is preferred or not to the lottery 'Wait-n-See' depends on Gentle's cost of carrying the umbrella, his risk aversion and his subjective probability p. Some argue that Gentle always can form subjective probabilities $(p, 1-p)$ also when he has no knowledge about the city area. However, being new in the city area and lacking knowledge about p, he may be unwilling to trust or 'gamble' with subjective probabilities and face the irreversible and unmeasurable uncertainty p, as did subjects show in Ellsberg's

experiment.

	Prob Dry		Prob Wet	Prob Sum
Rain	*No (0.5)*	*Yes (0.5)*		1
Shop	-	*Yes (p)*	*No (1-p)*	1
Buy Now	0.5	0.5	0	1
Wait-n-See	0.5	0.5p	0.5(1-p)	1

Table 1: Probabilities and of Dry and Wet Outcomes

Gentle may reason as follows: I choose 'Buy Now' and thereby avoid exposing myself and Mrs Gentle to the uncertainty $(p, 1-p)$ in 'Wait-n-See', which also involves the irreversible worst-case outcome that there should be no shops ($p = 0$) selling umbrellas in the city area. A reasoning coming to this choice, due to the difficulty of assigning subjective probabilities, involves aversion not just to risk but also aversion to ambiguity - ambiguity aversion - and Gentle's reasoning behind the choice demonstrates the Ellsberg's paradox. Unwilling to form subjective prior, and choosing 'Buy Now', Gentle avoids putting into effect the irreversible uncertainty $(p, 1-p)$ that follows 'Wait-n-See'.

When it comes to causes of ambiguity aversion, Ellsberg's setup has been repeated several times, supporting ambiguity aversion. In e.g. Fox and Tversky (1995) subjects were asked for their willingness to pay, resulting in much higher willingness to pay for the urn with known probabilities than for the ambiguous urn. However, ambiguity aversion disappeared in an experiment in which the two urns were evaluated in isolation, suggesting that the comparison of known vs unknown probabilities matters. Other experiments by e.g. Curley et al. (1986) showed that fear of negative evaluation when others observe the choice and may judge the decision-maker for it, increases his ambiguity aversion, which gives it a connection to how social norms may affect policymaking, see also Trautmann et al. (2008).[1] Even if we take these factors into account in Gentle's utility, expected utility theory and risk analysis would still be defective as long as he, due to lack of knowledge, is either unable or unwilling to assign subjective probabilities to outcomes. Ambiguity aversion is also closely related to the 'precautionary principle' which has been raised in e.g. the Rio Declaration article 15.[2] From a normative standpoint, 'Wait-n-See' may be an irresponsible choice by Gentle as it does not contain the precaution necessary to avoid exposing himself and Mrs Gentle to the irreversible uncertainty $(p, 1-p)$.

One of the most influential ways to model ambiguity aversion is by Gilboa and Schmeidler (1989) who formulated a maximin expected decision criterion, by weakening Savage's

[1]Frisch and Baron (1988) suggest that issues of blame, responsibility and regret, that observers might know more than the decision-maker, that a series of ambiguous gambles is more risky than a series of risky (non-ambiguous) games as possible explanations for ambiguity aversion in situations of ambiguity when others observe the choice.

[2]*'Where there are threats of serious or irreversible damage, lack of full scientific certainty shall not be used as a reason for postponing cost-effective measures to prevent environmental degradation.'* Ulph and Ulph (1997) have put forward that the benefits of insuring against irreversibility effects by actions now should balanced to the benefit of awaiting better scientific information by delaying action.

Sure-Thing Principle.[3] The decision-maker faces a set of probability distributions, and maximizes expected utility under the belief that the worst-case probability distribution is true, which in effect implies to assign more importance to bad outcomes in the reasoning. Kahneman and Tversky (1979) also found empirical evidence that individuals tend to assign more weight to low-probability extreme outcomes than would be implied by expected utility. The idea behind the maximin decision criterion can be illustrated as viewing Table 1 as a two-player zero-sum game between Gentle, choosing between rows, and 'Evil Nature', as a hypothetical minimizer, choosing between columns minimizing Gentle's utility. Given Gentle's belief that 'Evil Nature' will minimize his outcome by choosing 'Rain' and 'No Shops', his expected utility of the certain 'Buy Now' choice gets a relatively greater weight, possibly violating the Sure-Thing principle. Still we could find a subjective prior *ex post* that corresponds to Gentle's choice, but this does not imply that his choice can be explained by expected utility theory and pure risk aversion, as the reasoning guiding him to the choice may depend on ambiguity aversion or precautionary principles. The maximin decision criterion, as modeling fears of imperfect measurement of uncertainty, has been applied before in static models with the general result that it leads to an increase in abatement effort, e.g. Chichilnisky (2000) and Bretteville Froyn (2005), as well as dynamic models using a robust control approach with applications to e.g. water management Roseta-Palma and Xepapadeas (2004), climate change Hennlock (2008*b*) and biodiversity management Vardas and Xepapadeas (2008). In climate change policy, the maximin criterion may be used to model uncertainty aversion and precautionary concerns when the decision-maker has doubts about imperfect measurements of uncertainty in climate modeling and impact damage outcomes. However, a climate policy based on the worst imaginable worst-case belief (compare to the Dismal theorem) would likely be irresponsible when policy actions are connected to large expected costs of action. Thus, in an IAM we should therefore allow for a 'hypothetical minimizer' to choose over a range of worst-case beliefs corresponding to different degrees of ambiguity aversion and find statistically reasonable levels of worst-case beliefs.

1.1 Risk and Uncertainty in IAMs

The simplest way to introduce 'risk' in the literature on IAMs has been the so-called 'sensitivity-analysis approach'. Uncertain parameters are varied and values of carbon cost, optimal policy and outcomes are computed from several runs. This 'deterministic approach' becomes more sophisticated by replacing uncertain input parameter values by samplings from probability distributions and then obtain policy variables, expected benefits and costs as probability distributions from which mean and variance can be calculated. Also in deterministic models, the probability distributions for policies, costs and benefits can differ significantly from assumed probability distributions for input parameters, but this merely reflects that these variables are nonlinear functions of input parameters. Two early examples based on extensions of DICE (Nordhaus, 1992) resulted in 2 to 4 times higher carbon cost than the certainty case, reflecting the benefit of reducing risk of high future climate change costs, see Schauer (1995) and Nordhaus and Popp (1997). Another example mod-

[3] The Choquet expected utility (CEU) model by Schmeidler (1989) is another example.

eled catastrophic events by altering the probability distributions of damages as temperature increases (Nordhaus and Boyer, 2000). Nordhaus and Popp (1997) also imposed expected utility maximization and found carbon costs slightly higher than without maximization of expected utility, reflecting risk aversion. The Stern Review Stern (2007) performed a study using PAGE2002 (Hope, 2003) where several parameters are represented as probability distributions, to explore consequences of e.g. high climate sensitivities of 2.4^oC - 5.4^oC for the $5-95\%$ interval. Clearly, the optimal reductions in CO_2 emissions would differ largely whether the decision is based on e.g. the lower (2.0^oC) or upper (4.5^oC) level of the IPCC *likely* range, or a level outside the *likely* range (IPCC, 2007*a*). My main point (as model builder) in this paper is to be silent about this level and leave the question unanswered by letting the household face imperfect measurement of uncertainty over this range rather than one or another model builder's sometimes ad hoc guesses about its probability distributions as seen so far in IAM.

When it comes to imperfect measurements of uncertainty in IAM, Hennlock (2008*b*) introduced deeper uncertainty in a IAM approach using robust control in climate-economy modeling. Weitzman independently introduced deeper uncertainty, first in a draft to his Review of the Stern Report, and then in an early working paper of Weitzman (2008). Based on deeper uncertainty, the main results of Hennlock (2008*b*) and Weitzman (2008) seemed to tell the same story - uncertain probability distributions can justify large measures taken. In Hennlock (2008*b*) results emerged as a 'shadow ambiguity premium', inducing an ambiguity averse policymaker to take stringent measures (robust carbon pricing). Proposition 6 in Hennlock (2008*b*) showed that when a policymaker expresses *perfect* ambiguity aversion his expected shadow carbon cost becomes infinite, and hence, he 'backstop acts' by cutting carbon-generating production to zero. Weitzman's analysis, based on a static linear relationship between a utility function and a parameter with unknown probability distribution, showed that with Bayesian learning in a two-period analysis the result may be an infinite expected marginal utility at zero consumption levels (Weitzman, 2008).[4] In a model with multiple regions, Hennlock introduced deeper uncertainty also in regional damage functions besides climate sensitivity and found an increased stringency from ambiguity aversion as the worst-case beliefs about local damage become dependent on the worst-case beliefs about global climate sensitivity.[5]

In this theoretical paper we apply the IAM approach in Hennlock (2008*b*) - Analytical Model Uncertainty in an Integrated Climate-Economy with profit-maximizing firms (AMUICE-P) - but instead with a utility-maximizing household (AMUICE-C) in a model based on a two-sectoral extension of the DICE model in continuous time. We distort the mean of the climate sensitivity probability distribution and end up with a continuum of climate sensitivity probability distributions over an arbitrarily large (but finite) range so that they can cover e.g. the IPCC uncertainty range 2.0^oC - 4.5^oC (or an even greater range) that our (possibly ambiguity averse) household is willing to imagine (IPCC, 2007*a*). Given these multiple mean distorted probability distributions, which are understood as multiple

[4] Nordhaus (2009) comments on Weitzman (2008) and how the the result can depend on fat tails in the (posterior) probability distribution.

[5] M. Hennlock, A Robust Abatement Policy in a Climate Change Policy Model, Department of Economic and Statistics, University of Gothenburg, 2006, unpublished draft.

priors that our household can form about climate sensitivity, the household uses robust control to identify a robust policy design that works well over a range of climate sensitivity outcomes. The purpose of this first theoretical paper is not to perform a simulation or a sensitivity analysis, but to present an IAM and its analytically tractable solution using dynamic programming and to introduce deeper uncertainty and ambiguity aversion, modeled as maximin decision criterion within robust control, and finally comment on major consequences in connection to the discussion following the Stern Review on discounting and the Dismal Theorem. For a straightforward illustration of the approach, we only focus on uncertainty in equilibrium climate sensitivity in this paper which then has the following organization: Section 2 presents main features of the household's Ramsey alike problem in the deterministic problem. Section 3 presents how imperfect measurement of uncertainty and ambiguity aversion are introduced. Section 4 discusses the major outcomes of the analytical solution which is followed by a summary in section 5. The appendix contains the analytical solution.

2 The Climate-Economy Model

The AMUICE-C model has its next of kin in DICE when it comes to the way it captures economic and climatic phenomena. The major choice for the representative household is whether to consume a final good, to invest in productive capitals, or to slow global climate change by abatement and investing in carbon-neutral and (more efficient) carbon-intensive technology. Besides the two-sectoral approach, the major difference is the introduction of deeper uncertainty in climate sensitivity as one of the essential components in climate modeling that is subject to imperfect measurement of uncertainty (Roe and Baker, 2007). However, in this section we start by defining the deterministic household problem and introduce uncertainty in section 3.

The representative household problem is described as a Ramsey alike problem as in DICE but the representative household owns a carbon-intensive production sector and a carbon-neutral production sector (using a natural capital stock as input) with endogenous technology growth, inspired by Romer (1990), in both sectors, and a climate model of the type used in DICE in continuous time. The final good is composed by carbon-intensive input consumption C_t and carbon-neutral consumption G_t produced in the carbon-intensive and the carbon-neutral production sector, respectively. A CES function with constant elasticity of substitution σ and share parameter $\omega \in [0,1]$, describes how the goods C_t and G_t compose the final good. The objective function is taken from Sterner and Persson (2008):[6]

$$\max_{C,G,q,s,r} \int_0^\infty \frac{1}{1-\eta} \left[(1-\omega) C_t^{\frac{\sigma-1}{\sigma}} + \omega G_t^{\frac{\sigma-1}{\sigma}} \right]^{\frac{(1-\eta)\sigma}{\sigma-1}} e^{-\rho t} dt \tag{1}$$

with elasticity of marginal utility of consumption (constant relative risk aversion) η from consuming the final good where a high value of η is usually interpreted as high risk aversion or inequality aversion. The household maximizes objective (1) subject to the dynamic

[6] See Hoel and Sterner (2007) for a discussion on how the CES function affects the so-called Ramsey-rule.

system:

$$dK = \Big[(\upsilon_K + (1-r_t)A_{Kt}^{\tau})K_t^{\alpha}L_t^{1-\alpha} - cq_t^2 - C_t - \delta K_t\Big]\,dt \tag{2}$$

$$dA_K = \Big[\nu(r_{jt}Y_{Kjt})^{\tau}A_{Kt}^{1-\tau} - \delta_K A_{Kt}\Big]\,dt \tag{3}$$

$$dE_i = \left[(\upsilon_E + (1-s_t)A_t^{\psi})E_t^{\phi} - \frac{1}{\kappa}E_t - \Phi(T_t - T_0)E_t^{\phi} - \pi G_t\right]dt \tag{4}$$

$$dA_E = \Big[\beta(r_tY_{Ejt})^{\psi}A_{Et}^{1-\psi} - \delta_E A_{Et}\Big]\,dt \tag{5}$$

$$dM = \Big[\epsilon\varphi K_t^{\alpha}L_t^{1-\alpha} - \mu q_t - \Omega M_t\Big]\,dt \tag{6}$$

$$R_t = \frac{\lambda_0 ln(M_t/M_0)}{ln(2)} \tag{7}$$

$$dT = \frac{1}{\tau_1}\left(R_t + O_t dt - \lambda_1 T dt - \frac{\tau_3}{\tau_2}(T_t - \tilde{T}_t)dt\right) \tag{8}$$

$$d\tilde{T} = \frac{1}{\tau_3}\left(\frac{\tau_3}{\tau_2}(T_t - \tilde{T}_t)\right)dt \tag{9}$$

The major choice for the representative household is whether to consume the final good, to invest in productive capitals, or to slow global climate change by using the following policy variables: reduce the share of carbon-intensive composition in the final good by altering carbon-intensive share in consumption $C_t/(C_t + G_t)$, abatement effort q_t, research effort r_t in carbon-intensive research sector (more output for given emissions levels) and research effort s_t in carbon-neutral research sector. Sections 2.1 - 2.3 further describe the details of the dynamic programming problem (1) - (9). A complete list of all 32 model parameters is found in appendix A.2.

2.1 Carbon-Intensive Production Sector in (2) - (3)

The carbon-intensive production sector is described by the carbon-intensive capital growth equation (2) and the endogenous carbon-intensive technology growth equation (3). The carbon-intensive consumption good is produced by using carbon-intensive capital K_t, whose accumulation (2) is determined by production $A_{Kt}^{\tau}K_t^{\alpha}L_t^{1-\alpha}$ minus research expenditure $r_tA_{Kt}^{\tau}K_t^{\alpha}L_t^{1-\alpha}$ with $r_t \in [0,1]$, consumption of carbon-intensive good C_t and abatement cost. Applying the polluter-pays-principle, the carbon-intensive sector pays for abatement

effort q_t in (2) with a quadratic cost function due to capacity constraints as more effort is employed.

Carbon-intensive technology A_K develops endogenously in (3) with research effort $r_t \in [0,1]$ and the stock of abatement knowledge A_K as inputs in the research process. Thus a representative household whose research sector has generated many ideas in its history also has an advantage in generating new ideas relative to research sectors in less developed regions, see Romer (1990). The 'Malthusian constraints' $0 < \tau < 1$ and $\upsilon_K > 0$ in (2) 'stabilize' the dynamics as restrictions on future technology and capital sets such that carbon-intensive growth cannot 'go on for ever'. The same restriction in (3) also suggests that it requires more than a doubling researchers in order to double the number of ideas (as researchers may come up with the same ideas). The implementation of new discoveries in the production process, implies that some of the old knowledge cannot be used in the current production process. For example, some of artisans' knowledge before the industrial revolution was lost. Imperfect substitution of knowledge over time is reflected by $\delta_K \geq 0$.

2.2 Carbon-Neutral Production Sector in (4) - (5)

Growth equations (4) and (5) describe the dynamics of the carbon-neutral sector. The carbon-neutral consumption good G_t is produced by using carbon-neutral (environmental or natural) capital E_t whose accumulation follows (4). The first two terms in (4) describe a technology-enhanced natural growth function with carrying-capacity $\bar{E} = (\upsilon_K + \kappa(1 - s_t)A_{Et}^{\psi})^{1/(1-\phi)}$. Carbon-neutral technology A_E develops endogenously as in (3) and improves carbon-neutral capital growth (and raises carrying-capacity), thus counteracting the damage from temperature increases in (4). The 'Malthusian constraints' $0 < \psi < 1$ and $\upsilon_E > 0$ in (4) put restrictions on future technology and capital sets such that carbon-neutral growth cannot go on forever.

An overview of climate change impacts is found in IPCC (2001) and IPCC (2007*b*). Considered impacts are often on natural capitals; agriculture, forestry, water resources, loss of dry- and wetland (due to sea-level rise) etc. We let natural capital be damaged by an 'increasing-damage-to-scale' Cobb-Douglas function $\Phi(T_t - T_0)E_t^{\phi}$ in (4) adopted from Hennlock (2005) and Hennlock (2008*a*) with Φ as a climate impact parameter.[7] The 'increasing-damage-to-scale' implies that a given temperature increase leads to a greater total damage (or gain for $\Phi < 0$) the greater is the natural capital stock. The net carrying-capacity with climate impact is then:

$$\bar{E} = \left[\upsilon_E + \kappa\left(1 - s_t\right) A_{Et}^{\psi} - \Phi(T_t - T_0)\right]^{\frac{1}{1-\phi}} \tag{10}$$

Carbon-neutral technology A_{Et} can then also be seen as adaption technology in (10).

[7]Solutions are possible also when letting physical capital carry impact of climate change. However, separating stocks to damaging (physical) capital and damaged (environmental) capital, makes a unique solution possible corresponding to the verified value function.

2.3 Climate Modeling in (6) - (9)

Equations (6) - (9) describe a continuous-time modified version of the climate model used in DICE.[8] Total emissions in the first term in (6) is proportional to carbon-intensive production fraction and thus A_K increase output for given emissions flow where $\varphi > 0$. The second term is abatement level μq_t where $\mu > 0$. Net emissions flow accumulates to the global atmospheric CO_2 stock, M_t where $\epsilon > 0$ is the marginal atmospheric retention ratio and $\Omega > 0$ the rate of assimilation. The atmospheric CO_2 stock, M_t influences global mean atmospheric temperature T_t via the change in radiative forcing R_t (Wm^{-2}) in (7) which affects the energy balance of the climate system, and hence, the global mean atmospheric temperature T_t in (8) via the deep ocean temperature $\tilde{T}_t$ in (9).[9] The parameter λ_0 is essential for equilibrium climate sensitivity, τ_1 is the thermal capacity of atmosphere and upper ocean and τ_3 is the thermal capacity of deep ocean. $1/\tau_2$ is the transfer rate from the atmosphere and upper ocean layer to the deep ocean layer.[10]

3 Robust Control in Climate Policy Design

Robust control is a condition of analysis when specifications of the dynamics, in our case the climate model and climate impacts, are open to doubt by the decision-maker due to imperfect measurement of uncertainty. For illustrative purposes we here only introduce deeper uncertainty in climate sensitivity, though it could also be introduced in climate impacts.[11] In temperature equation (8) there are two possible places to introduce uncertainty in probabilities over climate sensitivity outcomes - via the radiative forcing parameter λ_0 in (7) and via the climate feedback parameter λ_1 reflecting uncertainty in the underlying physical processes. Both are conclusive for equilibrium climate sensitivity in (8) and introducing uncertain probability distributions in both λ_0 and λ_1 resulted in a solution with multiple solutions.[12] For illustrative purposes, we here want a straightforward unique solution and look at a household that only forms multiple priors about *equilibrium* climate sensitivity.

[8]The climate model was originally based on Schneider and Thompson (1981).

[9]For analytical tractability of the Isaacs-Bellman-Flemming equation, we approximate (7) in (8) by a square-root approximation

$$R_t \simeq \Lambda_0 \frac{\sigma_v \sqrt{M_t/M_0}}{\sqrt{2}\gamma} + \hat{\Lambda}_0 \frac{M_t/M_0}{2\gamma} \tag{11}$$

where γ is calibrated to fit (7). The corresponding change in equilibrium mean temperature $(\Lambda_0 + \hat{\Lambda}_0)/(\gamma\Lambda_1)$ in (8) from $M/M_0 = 2$ can be calibrated to follow (7).

[10]The geophysical parameter values used in the discrete DICE climate model are $\Lambda_0 = 4.1$, $\Lambda_1 = 1.41$, $1/\tau_1 = 0.226$, $\tau_3/\tau_2 = 0.44$ and $1/\tau_2 = 0.02$ and $\Omega = 0.0083$. For a calibration of these parameters to continuous form see e.g. Smirnov (2005).

[11]Knightian uncertainty in both climate sensitivity and local climate impacts was introduced in M. Hennlock, A Robust Abatement Policy in a Climate Change Policy Model, Department of Economic and Statistics, University of Gothenburg, 2006 unpublished draft, and resulted in significantly higher expected carbon cost for a given degree of ambiguity aversion as expected local damage becomes a function of worst-case beliefs in both local climate impact and global climate sensitivity.

[12]M. Hennlock, A Robust Abatement Policy in a Climate Change Policy Model, Department of Economic and Statistics, University of Gothenburg, 2006, unpublished draft.

Thus in what follows we let λ_0 capture all uncertainty in *equilibrium* climate sensitivity, although its uncertainty also has transitional feedback sources in λ_1 as analyzed by Roe and Baker (2007), but also their analysis is performed in equilibrium terms. We follow Hennlock (2008*b*) and define the following process:

$$B_{0t} = \hat{B}_{0t} + \int_0^t \Lambda_{0s} ds \quad \Lambda_{0s} \in [\Lambda_{0,min}, \Lambda_{0,max}] \tag{12}$$

where $d\hat{B}_0$ is the increment of the Wiener process $\hat{B}_{0t}$ on the probability space (Ξ_G, Φ_G, G) with variance $\sigma_v^2 \geq 0$ where $\{\hat{B}_{0t} : t \geq 0\}$. Moreover, $\{\Lambda_{0t} : t \geq 0\}$ is a progressively measurable drift distortion, implying that the probability distribution of B_{0t} itself is distorted and the probability measure G_0 is replaced by another unknown probability measure Q_0 on the space (Ξ_G, Φ_G, Q). The sensitivity process Λ_{0t} is then introduced in temperature equation (8) in the following way

$$\begin{aligned} dT = \frac{1}{\tau_1}\Bigg((\Lambda_{0t} dt + d\hat{B}_0) \frac{\sigma_v \sqrt{M_t/M_0}}{\gamma\sqrt{2}} + \frac{1}{2}\frac{\hat{\Lambda}_0}{\gamma}\frac{M_t}{M_0} dt + O_t dt \\ -\hat{\lambda}_1 T dt - \frac{\tau_3}{\tau_2}(T_t - \tilde{T}_t) dt \Bigg) \end{aligned} \tag{13}$$

and hence, temperature equation (13) follows an analytically tractable Ito process. Since both mean and variance of drift term Λ_{0t} are uncertain, (12) yields different statistics (priors) of equilibrium climate sensitivity in (13) where the interval $[\Lambda_{0,min}, \Lambda_{0,max}]$ indicates the maximum model specification error, e.g. corresponding to the range of climate sensitivity outcomes that the household is willing to imagine. Setting $\sigma_0 = 0$ yields the the 'benchmark model' that the household regards as an approximation to an unknown and unspecified global climate system that generates the true data.

3.1 Ambiguity Aversion as a Dynamic Maximin Decision Criterion

Ambiguity aversion violates the Sure-Thing Principle by Savage (1954), which is essential for ensuring that conditional preferences are well-defined and consistent over time and also being a basis for Bayesian updating. We assume that a rational decision-maker instead updates her beliefs to new information by a time consistent rule derived from backward induction using a dynamic maximin decision criterion adopted from robust control (Hansen et al., 2001).[13] We border to the problem a hypothetical minimizer that resides in the head of our household making her to think 'what if the worst about climate sensitivity turns out to be true'. We then introduce an aversion to uncertainty with $1/\theta_0 \in [0, +\infty]$ assigning how much our household listens to her 'minimizing voice'. The maximin criterion, with expectation operator ε, then takes the following form

$$\sup_{C,G,q,r,s} \inf_{\Lambda_0} \varepsilon \int_0^\infty \frac{1}{1-\eta} \left[C_t^{\frac{\sigma-1}{\sigma}} + \omega G_t^{\frac{\sigma-1}{\sigma}} \right]^{\frac{(1-\eta)\sigma}{\sigma-1}} e^{-\rho t} dt + \theta_0 R(Q_0) \tag{14}$$

[13]While Gilboa and Schmeidler (1989) view ambiguity aversion as a minimization of the set of probability measures, Hansen et al. (2001) set a robust control problem and let its perturbations be interpreted as multiple priors in max-min expected utility theory. Epstein and M. (2001) provides another updating process.

which can be formulated as a zero-sum differential game between the household (maximizer) and the hypothetical minimizer choosing the worst-case climate sensitivity prior path for the household. The last term contains a Lagrangian multiplier θ_0 and the finite entropy (Kullback-Leibler distance) $R(Q_0)$ as a statistical measure of the distance between the benchmark climate sensitivity prior and the worst-case climate sensitivity priors, generated by $\{\Lambda_{0s}\}$, in what follows: Recall that the unknown process in (12) will unexpectedly change the probability distribution of B_{0t}, having probability measure Q_0, relative to the distribution of $\hat{B}_{0t}$ having measure G_0. The Kullback-Leibler distance between probability measure Q_0 and G_0 is then:

$$R(Q_0) = \int_0^{\infty} \varepsilon_{Q_0} \left(\frac{|\Lambda_{0s}|^2}{2} \right) e^{-\rho t} ds \tag{15}$$

As long as $R(Q_0) < \Theta_0$ in (14) is finite

$$Q_0 \left\{ \int_0^t |\Lambda_{0s}|^2 ds < \infty \right\} = 1 \tag{16}$$

which has the property that Q_0 is locally continuous with respect to G_0, implying that G_0 and Q_0 cannot be distinguished with finite data, and hence, probability distributions cannot be inferred by using current finite climate data. Statistically this mimics the situation that current climate data from underlying physical processes is not sufficient to predict climate sensitivity probability distributions with certainty in accordance with Roe and Baker (2007). Following Hansen and Sargent (2001), a maximin constraint problem as in (14) can be rewritten

$$\max_{C,G,q,r,s} \min_{\Lambda_{0i}} \varepsilon \int_0^{\infty} \left\{ \frac{1}{1-\eta} \left[(1-\omega) C_t^{\frac{\sigma-1}{\sigma}} + \omega G_t^{\frac{\sigma-1}{\sigma}} \right]^{\frac{(1-\eta)\sigma}{\sigma-1}} + \frac{\theta_0 \Lambda_{0t}^2}{2} \right\} e^{-\rho t} dt \tag{17}$$

where the quadratic term contains mean distortions Λ_{0t} and the minimization with respect to Λ_{0t} creates a lower (worst-case) boundary of the value function. The corresponding policy feedback rule vector $(C_t^*, G_t^*, q_t^*, r_t^*, s_t^*)$ from the household's expected maximization would then be robust to priors that the household could imagine within the range $[0, \Lambda_{0t}^*]$. Maximizing-minimizing objective (17) subject to[14]

$$dK = \left[(\upsilon_K + (1-r_t) A_{Kt}^{\tau}) K_t^{\alpha} - c q_t^2 - C - \delta K \right] dt \tag{18}$$

$$dA_K = \left[\nu (r_t Y_{Kjt})^{\tau} A_{Kt}^{1-\tau} - \delta_K A_{Kt} \right] dt \tag{19}$$

$$dE = \left[(\upsilon_E + (1-s_t) A_t^{\psi}) E_t^{\phi} - \frac{1}{\kappa} E_t - \Phi(T_t - T_0) E_t^{\psi} - \pi G_t \right] dt \tag{20}$$

[14] In order to simplify, labor stock L_t is here omitted, hereinafter defining K_t as the amount of capital per unit labor.

$$dA_E = \left[\beta(r_t Y_{Et})^{\psi} A_{Et}^{1-\psi} - \delta_E A_{Et}\right] dt \tag{21}$$

$$dM = \left[\epsilon\varphi K_t^{\alpha} - \mu q_t - \Omega M_t\right] dt \tag{22}$$

$$\begin{aligned} dT = \frac{1}{\tau_1}\Bigg((\Lambda_{0t}dt + d\hat{B}_0)\frac{\sigma_v\sqrt{M_t/M_0}}{\sqrt{2}\gamma} + \hat{\Lambda}_0\frac{M_t/M_0}{2\gamma}dt + O_t dt \\ -\hat{\lambda}_1 T dt - \frac{\tau_3}{\tau_2}(T_t - \tilde{T}_t)dt\Bigg) \end{aligned} \tag{23}$$

$$d\tilde{T} = \frac{1}{\tau_3}\left(\frac{\tau_3}{\tau_2}(T_t - \tilde{T}_t)\right) dt \tag{24}$$

defines the household's stochastic optimization problem (17) - (24), bordered with the hypothetical minimizer choosing the household's upper boundary beliefs about climate-sensitivity mean distortions.

4 An Analytically Tractable Solution

The maximin dynamic programming problem (17) - (24) is solved by forming its Isaacs-Bellman-Flemming (IBF) equation in (38). Finding an analytically tractable solution to (38) by 'guessing-and-verifying' is tedious and left for appendix A.1. In short, the procedure goes as follows: Taking the first-order conditions of (38) and rearranging yield robust policy feedback rules. In order to identify shadow prices and costs, a value function that solves the IBF-equation (38) needs to be identified by a guessing-and-verifying procedure. Once a value function is verified that solves (38) it can be differentiated with respect to state variables and so identify the shadow price and cost partial derivatives. An analytically tractable solution to (17) - (24) is possible by carefully specifying 6 of the 32 parameters in appendix A.2. and its corresponding value function is identified in appendix A.1. Since, the objective function in (17) is time autonomous, any robust policy feedback rule will be time consistent (Dockner et al., 2000). Moreover, certainty equivalence makes the variance distortions in (12) irrelevant, thus only mean distortions are relevant. Taking the first-order condition of the IBF-equation (38) with respect to the policy vector $(C_t^*, G_t^*, q_t^*, r_t^*, s_t^*)$ and rearranging, yield robust policy feedback rules (25) - (30) where the partial derivatives are *expected* shadow prices and costs, which in general are functions of state variables.

$$C^*(K(t)) = \left(\frac{1-\omega}{\frac{\partial W}{\partial K_t}}\right)^2 e^{-2\rho t} \tag{25}$$

$$G^*(E(t)) = \left(\frac{\omega}{\pi\frac{\partial W}{\partial E_t}}\right)^2 e^{-2\rho t} \tag{26}$$

$$q^*(K(t), M(t)) = -\frac{\epsilon\mu}{2c}\frac{\frac{\partial W}{\partial M_t}}{\frac{\partial W}{\partial K_t}} \geq 0 \tag{27}$$

$$r^*(A_K(t), K(t)) = \frac{A_{Kt}^{1-\tau}(\nu\tau)^{\frac{1}{1-\tau}}}{K_t^{\alpha}}\left(\frac{\frac{\partial W}{\partial A_{Kt}}}{\frac{\partial W}{\partial K_t}}\right)^{\frac{1}{1-\tau}} \in [0,1] \tag{28}$$

$$s^*(A_E(t), E(t)) = \frac{A_{Et}^{1-\psi}(\beta\psi)^{\frac{1}{1-\psi}}}{E_t^{\phi}}\left(\frac{\frac{\partial W}{\partial A_{Et}}}{\frac{\partial W}{\partial E_t}}\right)^{\frac{1}{1-\psi}} \in [0,1] \tag{29}$$

The carbon-intensive consumption feedback rule in (25) is determined by the shadow price of carbon-intensive capital $\partial W/\partial K_t$. The lower the shadow price, the greater is consumption. The carbon-neutral consumption rule (26) has the same structure but with the instantaneous price $\pi > 0$. The abatement rule in (27) is determined by the relative price of shadow carbon price $-\partial W/\partial M_t$ with respect to carbon-intensive capital $\partial W \partial K_t$ (due to polluter-pays-principle). Since $\partial W/\partial M_t \leq 0$ abatement will be positive. The research effort feedback rule in carbon-intensive technology in (28) reduces carbon-intensity in carbon-intensive sector by greater fuel efficiency etc. and is determined by the relative shadow price of carbon-intensive technology with respect to carbon-intensive capital. By symmetry, the relative shadow price of carbon-neutral technology with respect to carbon-neutral capital is conclusive for carbon-neutral research effort feedback rule s_t in (29).

Minimizing the IBF equation (38) with respect to Λ_0 gives the optimal feedback rule identifying the household's worst-case mean distortion path, $\Lambda_0^*(M_t, T_t)$ in terms of its ambiguity aversion $1/\theta_0$ and expected shadow cost of climate change $\partial W/\partial T$:

$$\Lambda_{0t}^*(M(t), T(t)) = -\frac{\partial W}{\partial T}\frac{\sigma_v\sqrt{M_t/M_0}e^{\rho t}}{\theta_0\tau_1\gamma\sqrt{2}} \geq 0 \quad \Lambda_{0t}^* \in [\Lambda_{0,min}, \Lambda_{0,max}] \tag{30}$$

The optimal feedback rule (30) shows how the household updates its upper boundary of the range of climate sensitivities in (12) as it observes changes in the CO_2 stock and the mean temperature and $[0, \Lambda_{0t}^*]$ 'stakes out the corners' of the basis used for policymaking. Why does the worst-case mean distortion path $\Lambda_{0t}^*(M_t, T_t)$ depend on atmospheric CO_2 concentration rate and mean temperature? The explanation is that ambiguity aversion makes the household concerned about misreading observed increases in CO_2 concentration rate and temperature as source to global warming and damage impact, respectively. Accordingly, the household's worst-case beliefs alter to increases in CO_2 concentration rate and mean temperature *as though* observed increases eventually will cause a greater increase in equilibrium temperature and damage impact than expected so far. As precaution, robust policy design becomes more responsive to changes in observed CO_2 concentration rate and mean temperature, and works as an insurance to avoid (if possible) irreversible uncertainty over high-temperature outcomes up to a degree that corresponds to the household's degree of ambiguity aversion.

4.1 Ambiguity Aversion and Discounting

From the analytical expression of shadow carbon cost in proposition 1 below we see that ambiguity aversion has a similar effect on carbon cost path as has a low pure rate of time preference, which makes ambiguity aversion another gadget in the discussion on discounting that took place in the reviews following the Stern Review e.g. Nordhaus (2006), Dasgupta (2006), Weitzman (2007) and Sterner and Persson (2008).[15]

Proposition 1 *The expected shadow carbon cost corresponding to the housheold's robust control problem in (38) is:*

$$\frac{-\hat{\Lambda}_0 \dfrac{\omega\Phi\sqrt{\frac{2/\pi}{\rho+\frac{1}{2\kappa}+\frac{\beta^2}{8(2\rho+\delta_E)^2}}}}{\rho+\frac{\hat{\lambda}_1}{\tau_1}+\frac{\tau_3}{\tau_1\tau_2}\left(1-\frac{1}{1+\rho\tau_2}\right)} - \dfrac{\hat{\sigma}_v^2}{2\theta_0\tau_1\gamma}\left(\dfrac{\omega\Phi\sqrt{\frac{2/\pi}{\rho+\frac{1}{2\kappa}+\frac{\beta^2}{8(2\rho+\delta_E)^2}}}}{\rho+\frac{\hat{\lambda}_1}{\tau_1}+\frac{\tau_3}{\tau_1\tau_2}\left(1-\frac{1}{1+\rho\tau_2}\right)}\right)^2}{2\tau_1\gamma(\rho+\Omega)M_0} \cdot e^{-\rho t} \tag{31}$$

Proof: Solving (38) by guessing-and-verifying and identifying $\partial W/\partial M_t$ by determining the undetermined coefficient in (57).

Social carbon cost largely depends on geophysical parameters in the climate model (22) - (24) as well as economic parameters.[16] Moreover, ambiguity aversion $1/\theta_0 \in [0, +\infty]$ increases carbon cost, resulting in more stringent policy feedback rules. With no ambiguity aversion $\theta_0 \to +\infty$, and the effect of the quadratic term in (31) cancels and carbon cost and robust controls collapse to a certainty equivalent optimal control problem, using the benchmark climate sensitivity as basis in policymaking. In the other extreme, under perfect ambiguity aversion $\theta_0 \to 0$, the household takes into account 'uncut' worst-case mean distortions and expected carbon cost becomes infinite. Its consequences are further discussed in proposition 2 and 3 in section 4.2.

Even though patience ρ and ambiguity aversion $1/\theta_0$ affect carbon cost in a similar manner in (31), ambiguity aversion has an additional effect on policy compared to low utility discounting; it makes worst-case beliefs about equilibrium climate sensitivity responsive to changes in climate data observations over time as seen in (30) and how this in turn will affect policy depends *inter alia* on the damage function in (20). In (31) the carbon cost path is falling over time. The explanation is the way temperature deviation $T_t - T_0$ enters the Cobb-Douglas damage function, which in the IBF-equation (38) becomes

$$-\frac{\partial W}{\partial E_t}\Phi(T_t - T_0)E_t^{\phi} \tag{32}$$

Normally the increase in scarcity price $\partial W/\partial E_t$, as E_t falls from a temperature increase, would increase total damage in (32), however, the reduction in capital E_t also reduces total damage in (32). In the guessed-and-verified solution to (38) this reduction in total damage

[15] A discussion on discounting and uncertainty is also found in Guo et al. (2006).

[16] See appendix A.2. for the list of parameters.

gets the same rate as the increase in scarcity price, and thus, the two effects cancel each other and (32) becomes:

$$-\frac{\omega}{2}\sqrt{\frac{2/\pi}{\rho+\frac{1}{2\kappa}-\frac{\beta^2}{32(\rho+\delta_E/2)^2}}}\Phi(T_t-T_0) \tag{33}$$

which is linear in $T_t - T_0$. A non-linear formulation of temperature would call for a value function with a non-linear term in temperature which, in turn, would result in a shadow carbon cost being a function of M, T and $\tilde{T}$, which would also make policy directly responsive to observed changes in carbon concentration rate and temperatures via (31). Imposing a quadratic temperature term jointly with the nonlinear differential equation system for K_t, A_{Kt}, E_t and A_{Et} results in demanding calculations and is left out from this article to keep the illustration of analytical tractability straightforward.

4.2 Ambiguity Aversion and the Dismal Theorem

Proposition 6 in Hennlock (2008*b*) showed that a policymaker, who expresses *perfect* ambiguity aversion, exhibits infinite expected carbon cost and resorts to zero carbon-intensive production as precaution. In this two-sector consumer model, perfect ambiguity aversion results in a complete shift from carbon-intensive consumption to carbon-neutral consumption.

Proposition 2 *Let the household express perfect ambiguity aversion $\theta_0 \to 0$. Then its expected shadow cost of carbon-intensive capital $\partial W/\partial K_t \to +\infty$, resulting in the robust carbon-intensive consumption feedback rule*

$$\epsilon\left[\lim_{\theta_0\to 0} C^*(K^*(t)) = 0\right] \tag{34}$$

Proof: Setting $\theta_0 = 0$ in (57) gives $\lim_{\theta_0\to 0} f \to -\infty$ and $\lim_{\theta_0\to 0} a \to +\infty$ in (53). Differentiating (39) with respect to K_t gives $\lim_{\theta_0\to 0} \partial W/\partial K_t \to +\infty$ which in (25) yields (34).

Proposition 2 reminds us about the Dismal Theorem by Weitzman (2008), based on a static relationship between a utility function and a 'climate-sensitivity parameter' with unknown probability distribution, it demonstrated an infinite expected marginal utility at zero consumption level, if we believe (possibly as a result of ambiguity aversion) in fat-tail-priors. In my view the dismal theorem, rather than anything else, highlights the importance of taking ambiguity aversion seriously in climate change policy. Ambiguity aversion pushes the incentive for taking policy measures today to avoid (if possible) high-impact outcomes with low but uncertain probability magnitudes. If the household's $\theta_0 \to 0$, its beliefs embrace 'Weitzmanian fat-tails' about worst-case mean distortions in climate sensitivity and its expected shadow carbon cost in (31) becomes infinite. An important difference here is that there is (Bayesian) learning in Weitzman (2008) while there is no learning either in our model or in Hennlock (2008*b*).[17] However, the learning in Weitzman's two-period

[17] Gollier et al. (2000) focus e.g. on learning and uses prudence to give an interpretation of the precautionary principle.

model is not *realized* until we are far away (200 years?) into the future by the arrival of the second period.[18] In our model, the consequence of 'Weitzmanian fat-tails' corresponds to zero initial θ_0-beliefs and raises the question what initial level of ambiguity aversion (or precaution) the household should have today in presence of current scientific uncertainty. A household can express a high ambiguity aversion but it does not need to be perfect as will be discussed in the next section.

Even an infinite shadow carbon cost *do not necessarily* imply, as one might think at a first glance, that robust abatement and investment in technology take upper corners.

Proposition 3 *Let the household express perfect ambiguity aversion* $\theta_0 \to 0$*. Then its expected shadow cost of carbon* $\lim_{\theta_0 \to 0} \partial W/\partial M_t \to -\infty$*, resulting in the robust abatement feedback rule*

$$\epsilon\left[\lim_{\theta_0 \to 0} q^*(t, K^*(t))\right] = \frac{\frac{\mu}{c}(K_t^*)^{1/2}}{-\frac{\varphi}{(\rho+\delta/2)} + \sqrt{\left(\frac{\varphi}{(\rho+\delta/2)}\right)^2 + \frac{2\mu^2}{c(\rho+\delta/2-\frac{\nu^2}{32(\rho+\delta_K/2)})}}} \geq 0 \tag{35}$$

and the robust investment expenditure $r^*Y_t^*(K_t)$ *in carbon-intensive technology feedback rule*

$$\epsilon\left[\lim_{\theta_0 \to 0} r^*Y_t^*(K_t^*)\right] = \left(\frac{1}{2\rho + \delta_K}\right)^2 \frac{\nu^2}{4} K_t^* \geq 0 \tag{36}$$

Proof: Setting $\theta_0 = 0$ in (57) gives $\lim_{\theta_0 \to 0} f \to -\infty$ and $\lim_{\theta_0 \to 0} a \to +\infty$ in (53) and $\lim_{\theta_0 \to 0} b \to +\infty$ in (54). Differentiating (39) with respect to M_t, K_t and A_{Kt} yield $\partial W/\partial K_t \to +\infty$, $\partial W/\partial M_t \to -\infty$ and $\partial W/\partial A_{Kt} \to +\infty$ which in (27) and (28) reproduce (35) and (36).

Thus despite that both the shadow price of CO_2 stock and the shadow price of carbon-intensive capital stock explode - their ratio - the relative shadow price

$$\epsilon\left[\lim_{\theta_0 \to 0} -\frac{\frac{\partial W}{\partial M_t}}{\frac{\partial W}{\partial K_t}}\right] \tag{37}$$

converges to a finite positive value, and consequently, the robust abatement feedback rule converges to a maximum (but still finite) leverage for given capital paths in (35) also for 'uncut' worst-case mean distortions and an unbounded value function. When it comes to research effort in (36), both $\partial W/\partial K_t \to +\infty$ and $\partial W/\partial A_{Kt} \to +\infty$ but again the relative shadow price, conclusive for research effort, takes a finite value as seen in (36). The results in (35) and (36) are specific due to the model formulation of endogenous 'relative-shadow-price-driven' technology growths and that carbon-intensive capital pays for current and future abatement (polluter-pays-principle).

[18] Tol et. al. Yohe and Tol (2007) admit the importance of the dismal theorem but also put a label on it: *'Warning: Not to be taken to its logical extreme in application to real world problems.'*

4.3 How much Precaution is too Cautious?

Still, proposition 2 and 3, the Dismal Theorem and proposition 6 in Hennlock (2008*b*) bring up the question how much precaution is justified. Weitzman applies the value of statistical life (VSL) parameter to generate a lower boundary on consumption (Weitzman, 2008). We may take another approach, considering that a finite upper boundary on ambiguity aversion $1/\theta_0$, when interpreted as precaution, would cut the considered worst-case mean distortion and leave a minimized finite value of the value function. A possible upper boundary would be to set θ_0 sufficiently high (as an upper boundary for precaution) to make it difficult to statistically distinguish alternative worst-case climate sensitivity outcomes from a benchmark sensitivity properly set within the IPCC climate sensitivity range. But that brings us to a normative statement how much aversion to uncertainty should be involved in climate change policy for it to be consistent with a 'precautionary principle' as formulated by e.g. article 15 of the Rio Declaration? To get a feeling for what θ_0 levels we may talk about, we take some upper boundaries 4.5^oC and 6.0^oC as mentioned in IPCC (2007*a*) and derive corresponding lower boundaries for θ_0 by rearranging the optimal feedback rule (30). This suggests boundaries for θ_0 in the interval 1 - 2 percent which in (31) suggests a carbon cost path that is 3 to 5 times higher than the certainty case, compared to Nordhaus and Popp (1997) who got up to 4 times the certainty case due to pure risk aversion. Recall that our result (for simple tractability reason) uses an elasticity of marginal utility of only $\eta = 0.5$ and *linear* one-sector damage while Nordhaus and Popp (1997) use $\eta = 1$ and non-linear damage, suggesting that ambiguity aversion corresponding to the IPCC range up to 4.5^oC together with a higher η and non-linear damage could justify more stringency than what pure risk aversion has shown in IAMs so far. The effect of ambiguity aversion on carbon cost comes in addition and independently of the effect of a low utility discounting (here 0.03) as the latter increases both the certainty and the uncertainty cost by the same percentage. As a precursor, table 2 shows derived values for $\hat{\theta}_0$ and robust shadow carbon cost (as a shadow ambiguity premium) expressed as markups (times the certainty carbon cost path) for various boundaries of Λ_0, here against a benchmark of $\hat{\Lambda}_0$ corresponding to 1.5^oC.[19] The point with a statistical approach would be to calibrate θ_0 by calculating overall detection error probabilities $p(1/\theta_0)$ for distinguishing a benchmark climate model from worst-case climate models by varying $1/\theta_0$.[20] Ambiguity aversion then translates to what detection error probabilities we are willing to accept for distinguishing worst-case climate models from an approximative benchmark climate model. In the case science should narrow predictions of climate sensitivity in the future, the corresponding adjustment $\hat{\theta}_{0t}$ over time replaces learning by updating the range of climate-sensitivity outcomes used as basis for

[19]The calculations are based on a pure rate of time preference 0.03, elasticity of marginal utility $\eta = 0.5$, $\Phi = 0.10$ and $\omega = 0.5$. Geophysical parameter values are based on calibrations by the author as well as Smirnov (2005) for a continuous-time version of the climate model in DICE. The parameter values used are $\sigma_v = 1$, $\Lambda_0 = 3.38$, $\gamma = 0.5719$, $\lambda_1 = 1.41$, $1/\tau_1 = 0.226$, $\tau_3/\tau_2 = 0.44$ and $1/\tau_2 = 0.02$ and $\Omega = 0.0083$.

[20]Hansen and Sargent (2001) suggest that a robustness parameter θ should be set sufficiently high for it to take long time series to distinguish the benchmark model from worst-case models. By calculating likelihood ratio under benchmark and worst-case models Hansen and Sargent (2008) suggest calculating overall detection error probability using detection error probabilities conditional on each model, respectively. For $1/\theta_0 = 0$ models are identical and $p = 0.5$. In general the greater is $1/\theta_0$, the lower is then p.

Range (^oC)	$\hat{\theta}_0$	Shadow Ambiguity Premium
1.5 - 1.5	$+\infty$	1
1.5 - 3.0	0.039	2.31
1.5 - 4.5	0.019	3.62
1.5 - 6.0	0.013	4.93
1.5 - 7.0	0.011	5.81
1.5 - 8.0	0.009	6.68
1.5 - 9.0	0.0078	7.56
1.5 - 10.0	0.0068	8.43
1.5 - 20.0	0.0031	17.17

Table 2: Derivation of initial $\hat{\theta}_0$ and Shadow Ambiguity Premium

robust policy design. Precaution then becomes a function of waiting time for enough data to discriminate worst-case climate models from approximative benchmark climate models.

4.4 Tractability vs Complexity in Analytical IAM

A technical feature of our IAM is analytically tractable dynamic programming solutions in continuous time instead of computer-based numerical simulations in discrete time as usual seen in IAM. Analytically tractable solutions to non-linear differential games, using guessing-and-verifying methods in dynamic programming, are usually extremely difficult to identify. Still an analytical solution usually has better reliability, allows for deeper understandings as trajectories can be traced down to their explicit functional forms, and the model can also serve as a basis for which more complex extensions gradually can be added. To obtain a unique analytically tractable solution some major simplifications have been made e.g. (i) specifications have been carefully chosen for 6 of the 32 parameters (see appendix A.2.) that amongst other things make the objective function additively separable while remaining 26 parameters are free to be varied for sensitivity analysis and simulations, (ii) linear damage only in the non-carbon-generating sector (two-sector damage results in multiple and not straightforward analytically tractable solutions),[21] (iii) Knightian uncertainty only in the radiative forcing parameter, and (iv) a slightly modified temperature equation to make it follow an Ito process, but still it can be calibrated to follow the original temperature equation used in DICE closely.

5 Summary

Imperfect measurement of uncertainty in climate modeling and climate impact damage suggest that the traditional concepts of subjective expected utility theory and risk aversion may be insufficient in explaining and capturing reasonings behind precautionary principles in climate policy. We applied the robust control IAM approach in Hennlock (2008*b*) which

[21]M. Hennlock, A Robust Abatement Policy in a Climate Change Policy Model, Department of Economic and Statistics, University of Gothenburg, 2006, unpublished draft.

differs from existing 'risk analysis' in IAMs in that it does not fix one or another model builder's sometimes ad hoc (subjective) probability distributions to uncertain parameters. Statistically it mimics a decision-making process in which finite climate data from underlying physical processes is statistically insufficient to predict climate sensitivity probability distributions with certainty (IPCC, 2007*a*). We applied the approach to a two-sectoral extension of DICE - with a carbon-intensive and a carbon-neutral sector and a representative household that expresses ambiguity aversion and therefore uses robust control to identify robust policy feedback rules. Robust control is a condition of analysis when specifications of the dynamics, in our case climate modeling and climate impacts, are open to doubt by the decision-maker due to imperfect measurement of uncertainty. We found the following results:

1. Ambiguity aversion puts forward an incentive to take action via a 'shadow ambiguity premium' on social carbon cost, inducing a robust policy feedback design that avoids (if possible) a realization of high-impact low-probability outcomes.
2. As an upper boundary for precautionary beliefs, we proposed the statistical approach, that θ_0 is set sufficiently high to make it statistically difficult to distinguish worst-case models from benchmark models using detection error probabilities.
3. Ambiguity aversion in combination with linear damage, increases shadow carbon cost in a similar way as a low pure rate of time preference.
4. Ambiguity aversion in combination with non-linear damage would, besides even greater stringency, make policy feedback rules responsive to changes in climate data observations as it makes the household concerned about misreading increases in observed CO_2 concentration rate or temperature as sources to global warming and impact. This behavior cannot be calibrated by a low pure rate of time preference.
5. Perfect ambiguity aversion results in an infinite expected shadow carbon cost and zero carbon-intensive consumption. Still, the relative shadow price converges to a finite value and the robust abatement feedback rule is finite despite an unbounded value function.
6. Dynamic programming has identified an analytically tractable solution to this IAM.

There are several interesting ways to extend robust control in climate change policy analysis. Thinking about Roe and Baker (2007), who stated that it is not likely that scientific progress or learning will narrow the thicker high-temperature tail, strengthens the argument for a robust policy design as a complement to (Bayesian) learning and traditional 'risk analysis' in IAMs.[22] An interesting extension should be to combine ambiguity aversion and learning

[22] Roe and Baker (2007) do not therefore expect the range presented in the next IPCC report to be different from that in the 2007 report, *'we are constrained by the inevitable: the more likely a large warming is for a given forcing (i.e. the greater the positive feedbacks) the greater the uncertainty will be in the magnitude of that warming.'*

under imperfect measurement of uncertainty and balance benefits of insuring against irreversible *deeper uncertainty* by action to benefits of awaiting better scientific information by delaying action, as has already been done within risk analysis (see e.g. Gollier et al., 2000). A scientific-oriented extension is to develop and adapt statistical methods based on detection error probabilities for discriminating worst-case models from approximative benchmark models within IPPC ranges, as well as to simulate and calibrate this IAM for reasonable empirical parameter values. Other extensions are; non-linear damage which together with ambiguity aversion would make policy responsive to climate data observations as discussed in section 4.1, to extend to a game with multiple regional representative households involved in cap-and-trade as well as to repeat the dynamic maximin decision criterion to include several uncertain model parameters also in e.g. climate impact damage.[23]

Appendix

A.1. The Dynamic Programming Problem

The section presents a solution structure to AMUICE-C with a parameter-setting that allows for an analytically tractable solution to the zero-sum differential game defined by objective (17) and dynamic system (18) - (24). Forming the Isaacs-Bellman-Fleming (IBF) dynamic programming equation (see Fleming and Richel, 1975):

$$
\begin{aligned}
-\frac{\partial W}{\partial t} &= \qquad (38)\\
\max_{C,G,q,r,s}\min_{\Lambda_0} &\left\{\frac{1}{1-\eta}\left[(1-\omega)C_t^{\frac{\sigma-1}{\sigma}}+\omega G_t^{\frac{\sigma-1}{\sigma}}\right]^{\frac{(1-\eta)\sigma}{\sigma-1}}+\frac{\theta_0\Lambda_{0t}^2}{2}\right\}e^{-\rho t}\\
&+\frac{\partial W}{\partial K}\left[(\upsilon_K+(1-r_t)A_{Kt}^{\tau})K_t^{\alpha}-cq_t^2-C_t-\delta K_t\right]\\
&+\frac{\partial W}{\partial A_K}\left[\nu(s_tY_{Kt})^{\tau}A_{Kt}^{1-\tau}-\delta_K A_{Kt}\right]\\
&+\frac{\partial W}{\partial E_K}\left[(\upsilon_E+(1-s_t)A_t^{\Psi})E_t^{\phi}-\frac{1}{\kappa}E_t-\Phi(T_t-T_0)E_t^{\phi}-pG_t\right]\\
&+\frac{\partial W}{\partial A_E}\left[\beta(r_tY_{Et})^{\psi}A_{Et}^{1-\psi}-\delta_E A_{Et}\right]+\frac{\partial W}{\partial M}\left[\epsilon\varphi K_t^{\alpha}L_t^{1-\alpha}-\mu q_t-\Omega M_t\right]\\
&+\frac{\partial W}{\partial T}\frac{1}{\tau_1}\left[\Lambda_{0t}\frac{\sigma_v\sqrt{M_t/M_0}}{\sqrt{2}\gamma}+\hat{\Lambda}_0\frac{M_t/M_0}{2\gamma}+O_t-\hat{\lambda}_1Tdt-\frac{\tau_3}{\tau_2}(T_t-\tilde{T}_t)\right]\\
&+\frac{1}{2}\frac{\partial^2 W}{\partial T^2}\sigma_v^2M_t+\frac{\partial W}{\partial \tilde{T}}\left[\frac{1}{\tau_3}\left(\frac{\tau_3}{\tau_2}(T_t-\tilde{T}_t)\right)\right]
\end{aligned}
$$

The robust control vector $\Gamma_t^*=(C_t,G_t,q_t,r_t,s_t)$ is given by maximizing the partial differential equation (38) with respect to policy variables and minimizing with respect to Λ_{0t} and solving for feedback rules.

[23] M. Hennlock, A Robust Abatement Policy in a Climate Change Policy Model, Department of Economic and Statistics, University of Gothenburg, 2006, unpublished draft.

Proposition 4 *The value function* $W(K, A_K, E, A_E, M, T, \tilde{T}, t)$

$$= \left(aK^{1-\alpha} + bA_K^{\tau} + dE^{1-\phi} + eA_E^{\psi} + fM + gT + h\tilde{T} + k\right)e^{-\rho t} \tag{39}$$

satisfy the differential equation system formed by (38).

Proof: Substituting (25) to (30) into (38) and collecting terms forms the indirect Isaacs-Bellman-Fleming equation. Guessing the value function (39), taking FOC and rearranging give the optimal feedback rules. An analytically tractable solution is possible by setting $\sigma = 2$ and $\eta = \tau = \alpha = \phi = \psi = 1/2$ while the remaining 26 parameters, listed in appendix A.2., can be set free. The carbon-intensive consumption feedback rule is

$$C^*(K_t) = \frac{4(1-\omega)^2}{a^2} K_t \geq 0 \tag{40}$$

where a is defined in (53). The carbon-neutral consumption feedback rule is

$$G^*(E_t) = \frac{4\omega^2}{(\pi d)^2} E_t \geq 0 \tag{41}$$

where d is defined in (55). The abatement feedback rule is

$$q^*(K_t) = -\frac{\epsilon\mu}{c}\frac{f}{a} K_t^{1/2} \geq 0 \tag{42}$$

where a is defined in (53) and f in (57). The carbon-intensive research effort feedback rule is

$$r^*(K_t) = \left(\frac{b\nu}{2a}\right)^2 \left(\frac{K_t}{A_{Kt}}\right)^{1/2} \in [0, 1] \tag{43}$$

where b is defined in (54). The carbon-neutral research effort feedback rule is

$$s^*(E_t) = \left(\frac{e\beta}{2d}\right)^2 \left(\frac{E_t}{A_{Et}}\right)^{1/2} \in [0, 1] \tag{44}$$

where e is defined in (56). The minimizer's feedback rule is

$$\Lambda^*_{0t}(M(t)) = \frac{\frac{d}{2}\Phi}{\rho + \frac{\hat{\lambda}_1}{\tau_1} + \frac{\tau_3}{\tau_1\tau_2}(1 - \frac{1}{1+\rho\tau_2})} \frac{\sigma_v\sqrt{M_t/M_0}}{\theta_0\tau_1\gamma\sqrt{2}} \geq 0 \tag{45}$$

Using value function (39) and substituting (40) to (45) in (38) yield the equation system

$$\rho a = \frac{2(1-\omega)^2}{a} - a\frac{\delta}{2} + \frac{a\nu^2}{32(\rho + \delta_K/2)^2} + \frac{(f\epsilon\mu)^2}{2ac} + f\epsilon\varphi \tag{46}$$

$$\rho b = \frac{a}{2} L_K^{1/2} - b\frac{\delta_K}{2} \tag{47}$$

$$\rho d = \frac{2\omega^2}{\pi} + \frac{(e\beta)^2}{8d_i} - \frac{1}{2\kappa} \tag{48}$$

$$\rho e = \frac{d}{2} - e\frac{\delta_E}{2} \tag{49}$$

$$\rho f = -\frac{1}{2\theta_0 M_0}\left(\frac{g\sigma_v}{\tau_1\gamma\sqrt{2}}\right)^2 - f\Omega + \frac{g\hat{\Lambda}_0}{2\tau_1\gamma M_0} \tag{50}$$

$$\rho g = -\frac{d\Phi}{2} - g_i\frac{\hat{\Lambda}_1}{\tau_1} - g\frac{\tau_3}{\tau_1\tau_2} + h\frac{1}{\tau_2} \tag{51}$$

$$\rho h = g\frac{\tau_3}{\tau_1\tau_2} - h\frac{1}{\tau_2} \tag{52}$$

Solving the equation system (46) - (52) for undetermined coefficients gives the coefficients in terms of parameter values

$$\begin{aligned} a = {} & \frac{f\epsilon\varphi}{2\left(\rho + \frac{\delta}{2} - \frac{\nu^2}{32(\rho+\delta_K/2)^2}\right)} \\ & + \frac{1}{2}\sqrt{\left(\frac{f\epsilon\varphi}{\rho + \frac{\delta}{2} - \frac{\nu^2}{32(\rho+\delta_K/2)^2}}\right)^2 + \frac{8(1-\omega)^2 + 2\frac{(f\epsilon\mu)^2}{c}}{\rho + \frac{\delta}{2} - \frac{\nu^2}{32(\rho+\delta_K/2)^2}}} \end{aligned} \tag{53}$$

$$b = \frac{a}{2\rho + \delta_K} \tag{54}$$

$$d = \omega\sqrt{\frac{2/\pi}{\rho + \frac{1}{2\kappa} - \frac{\beta^2}{32(\rho+\delta_E/2)^2}}} \tag{55}$$

$$e = \frac{d}{2\rho + \delta_E} \tag{56}$$

$$f = \frac{g\left(\hat{\Lambda}_0 - \frac{g\sigma_v^2}{2\theta_{0i}\tau_1\gamma}\right)}{2\tau_1\gamma(\rho+\Omega)M_0} \tag{57}$$

$$g = \frac{-\frac{d}{2}\Phi}{\rho + \frac{\hat{\lambda}_1}{\tau_1} + \frac{\tau_3}{\tau_1\tau_2}(1 - \frac{1}{1+\rho\tau_2})} \tag{58}$$

$$h = g\frac{\tau_3}{\tau_1\tau_2(\rho + \frac{1}{\tau_2})} \tag{59}$$

The coefficients in (53) - (59) are uniquely defined, the coefficient k in proposition 4 is uniquely determined by (53) - (59), and hence, the feedback rules (40) to (45) corresponding to the guessed value function (39) are unique and the solution is verified. **Q.E.D.**

A.1.1. Transitional Dynamics

To find the optimal trajectories in the dynamic system, the feedback rules (40) to (45) are substituted in dynamic system (18) - (24) which then gives

$$\begin{aligned} dK = \Bigg[(\upsilon_K + A_{Kt}^{1/2})K_t^{1/2} - \left(\frac{b}{a}\right)^2 \frac{\nu^2}{4}K_t - c\left(-\frac{\epsilon\mu}{c}\frac{f}{a}K_t^{1/2}\right)^2 \\ -\frac{4(1-\omega)^2}{a^2}K_t - \delta K_t\Bigg]dt \end{aligned} \tag{60}$$

$$dA_K = \left[\nu\left(\left(\frac{b}{a}\right)^2\frac{\nu^2}{4}K_t\right)^{1/2} A_{Kt}^{1/2} - \delta_K A_{Kt}\right] dt \tag{61}$$

$$\begin{aligned} dE_i = \Bigg[(\upsilon_E + A_t^{1/2})E_t^{1/2} - \left(\frac{e}{d}\right)^2\frac{\beta^2}{4}E_t - \frac{1}{\kappa}E_t \\ -\Phi(T_t - T_0)E_t^{1/2} - \pi\frac{4\omega^2}{(\pi d)^2}E_t\Bigg]dt \end{aligned} \tag{62}$$

$$dA_E = \left[\beta\left(\left(\frac{e}{d}\right)^2\frac{\beta^2}{4}E_t\right)^{1/2} A_{Et}^{1/2} - \delta_E A_{Et}\right] dt \tag{63}$$

$$dM = \left[\epsilon\varphi K_t^{1/2}L_t^{1/2} + \mu\frac{\epsilon\mu}{c}\frac{f}{a}K_t^{1/2} - \Omega M_t\right] dt \tag{64}$$

$$\begin{aligned} dT = \frac{1}{\tau_1}\Bigg(\frac{\sigma_v\sqrt{M_t/M_0}}{\sqrt{2}\gamma}d\hat{B}_0 - \frac{g\sigma_v^2 M_t/M_0}{2\theta_0\tau_1\gamma^2}dt + \hat{\Lambda}_0\frac{M_t/M_0}{2\gamma}dt + O_t dt \\ -\hat{\Lambda}_1 T dt - \frac{\tau_3}{\tau_2}(T_t - \tilde{T}_t)dt\Bigg) \end{aligned} \tag{65}$$

$$d\tilde{T} = \frac{1}{\tau_3}\left(\frac{\tau_3}{\tau_2}(T_t - \tilde{T}_t)\right) dt \tag{66}$$

To simplify expressions, define the following parameters

$$\Pi_K \equiv \left(\frac{b}{a}\right)^2 \frac{\nu^2}{4} - c\left(-\frac{\epsilon\mu}{c}\frac{f}{a}\right)^2 - \frac{4(1-\omega)^2}{a^2} - \delta \quad \Gamma_K \equiv \left(\frac{b}{a}\right)^2 \frac{\nu^2}{2} \tag{67}$$

$$\Pi_E \equiv \left(\frac{e}{d}\right)^2 \frac{\beta^2}{4} E_t - \frac{1}{\kappa} - \pi\frac{4\omega^2}{(\pi d)^2} \quad \Gamma_E \equiv \left(\frac{e}{d}\right)^2 \frac{\beta^2}{2} \tag{68}$$

$$\Theta \equiv \epsilon\varphi + \mu\frac{\epsilon\mu}{c}\frac{f}{a} \quad \chi \equiv \frac{\hat{\Lambda}_0}{2\tau_1\gamma} \quad \Xi \equiv \frac{\hat{\Lambda}_1}{\tau_1} + \frac{\tau_3}{\tau_1\tau_2} \tag{69}$$

And the dynamic system (60) - (66) can be rewritten as

$$dK = \left[(\upsilon_K + A_{Kt}^{1/2})K_t^{1/2} - \Pi_K K_t\right] dt \tag{70}$$

$$dA_K = \left[\Gamma_K A_{Kt}^{1/2} K_t^{1/2} - \delta_K A_{Kt}\right] dt \tag{71}$$

$$dE = \left[(\upsilon_E + A_{Et}^{1/2})E_t^{1/2} - \Pi_E E_t - \Phi(T_t - T_0)E_t^{1/2}\right] dt \tag{72}$$

$$dA_E = \left[\Gamma_E A_{Et}^{1/2} E_t^{1/2} - \delta_E A_{Et}\right] dt \tag{73}$$

$$dM = \left[\Theta K_t^{1/2} - \Omega M_t\right] dt \tag{74}$$

$$dT = \chi M_t - \Xi T_t - \frac{\tau_3}{\tau_1\tau_2}\tilde{T} \tag{75}$$

$$d\tilde{T} = \frac{1}{\tau_2}\left(T_t - \tilde{T}_t\right) dt \tag{76}$$

Besides computer-based methods an analytical solution to (70) - (76) can be found by using transformations $\hat{K} \equiv K^{1/2}$, $\hat{A}_K \equiv A_K^{1/2}$, $\hat{E} \equiv E^{1/2}$ and $\hat{A}_E \equiv A_E^{1/2}$ transforming the system to a linear system.

A.1.2. Steady States

From (60) to (66) the steady states $(\bar{K}, \bar{A}_K, \bar{E}, \bar{A}_E, \bar{M}, \bar{T}, \bar{\tilde{T}})$ as $t \to \infty$ in the $4N+3$ state space can be derived in terms of parameter values. Applying certainty equivalence in (38) yields

$$\bar{K} = \left(\frac{\upsilon_K}{\Pi_K - \frac{\Gamma_K}{\delta_K}}\right)^2 \quad \bar{A}_K = \left(\frac{\frac{\Gamma_K}{\delta_K}}{\Pi_K - \frac{\Gamma_K}{\delta_K}}\right)^2 \tag{77}$$

$$\bar{E} = \left(\frac{\upsilon_E + \Phi(\bar{T}_t - T_0)}{\Pi_E - \frac{\Gamma_E}{\delta_E}} \right)^2 \quad \bar{A}_E = \left(\frac{\frac{\Gamma_E}{\delta_E}}{\Pi_E - \frac{\Gamma_E}{\delta_E}} \right)^2 \tag{78}$$

$$\bar{M} = \frac{\Theta}{\Omega} \frac{\upsilon_K}{\Pi_K - \frac{\Gamma_K}{\delta_K}} \quad \bar{T} = \frac{\chi}{\Xi} \frac{\Theta}{\Omega} \frac{\upsilon_K}{\Pi_K - \frac{\Gamma_K}{\delta_K}} \quad \bar{\tilde{T}} = \bar{T} \tag{79}$$

The corresponding steady state policy variables as $t \to \infty$ are found by substituting (77) - (79) in (40) - (44).

A.2. List of Parameters

The found analytically tractable solution required 6 of 32 parameters to be specified as below. The remaining 26 parameters below are free to be varied.

Free Parameters

$\rho > 0$	pure time preference
$\omega \in [0, 1]$	share parameter in objective function
$1/\theta_0 \in [0, +\infty]$	degree of ambiguity aversion
$c \geq 0$	abatement cost parameter
Θ	climate impact parameter
$\pi \geq 0$	relative price carbon-neutral input good
$\nu \geq 0$	carbon-intensive research sector efficiency parameter
$\beta \geq 0$	carbon-neutral research sector efficiency parameter
$\delta \geq 0$	depreciation rate carbon-intensive capital
$1/\kappa \geq 0$	depreciation rate carbon-neutral capital
$\delta_K \geq 0$	depreciation rate carbon-intensive technology
$\delta_A \geq 0$	depreciation rate carbon-neutral technology
$\varphi \geq 0$	carbon-intensity
$\mu \geq 0$	abatement effort efficiency
$\upsilon_K \geq 0$	Malthusian constraint carbon-intensive sector
$\upsilon_E \geq 0$	Malthusian constraint carbon-neutral sector

Free Parameters in the Climate Model

$\hat{\lambda}_0 \geq 0$	radiative forcing parameter
$\hat{\lambda}_1 \geq 0$	climate feedback parameter
$\tau_1 \geq 0$	thermal capacity of atmospheric layer
$\tau_3 \geq 0$	thermal capacity of deep ocean layer
$1/\tau_2 \geq 0$	transfer rate from the upper layer to the deeper ocean layer
$1/\Omega \geq 0$	transfer rate of CO_2 from atmosphere to other reservoirs
$\epsilon \geq 0$	marginal atmospheric retention ratio
$M_0 \geq 0$	initial CO_2 concentration rate
$T_0 \geq 0$	initial atmospheric mean temperature
$\tilde{T}_0 \geq 0$	initial deep ocean mean temperature

Specified Parameters in the Analytically Tractable Solution

$\sigma = 2$	elasticity of substitution
$\eta = 0.5$	elasticity of marginal utility of final good
$\alpha = 0.5$	capital intensity carbon-intensive production
$\phi = 0.5$	capital intensity carbon-neutral production
$\tau = 0.5$	Malthusian exponent constraint carbon-intensive sector
$\psi = 0.5$	Malthusian exponent constraint carbon-neutral sector

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